GV-SHEAVES, FOURIER-MUKAI TRANSFORM, AND GENERIC VANISHING

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Contents

1. Introduction

In this paper we use homological techniques to establish a general approach to generic vanishing theorems, a subject originated with the pioneering work of Green-Lazarsfeld [GL1] and [GL2]. Our work is inspired by a recent paper of Hacon [Hac]. Roughly speaking, we systematically investigate – in a general setting – the relation between three concepts: (1) generic vanishing of (hyper-)cohomology groups of sheaves (complexes) varying in a parameter space; (2) vanishing of cohomology *sheaves* of Fourier-Mukai transforms; (3) a certain vanishing condition for honest cohomology groups, related to the vanishing of higher derived images in the spirit of the Grauert-Riemenschneider theorem (which is a special case of this phenomenon). The relationship between these concepts establishes a connection between Generic Vanishing theory and the theory of Fourier-Mukai functors. One of the main points of the paper is that, for projective varieties with mild singularities, these three concepts are essentially equivalent and, more importantly, this equivalence holds not just for derived equivalences, but in fact for any *integral transform*. This principle produces a number of new generic vanishing results, which will be outlined below.

Let us briefly describe the three concepts above. For the sake of brevity we do this only in the special case of smooth varieties and locally free kernels (we refer to §2,3 for a complete treatment, in greater generality). Let X and Y be smooth projective varieties, and let P be a locally free sheaf on $X \times Y$. Here Y can be thought of as a moduli space of sheaves on X and P as the universal sheaf. Generic vanishing theorems for cohomology groups deal with the *cohomological support loci*

$$V^i_P(\mathcal{F}) = \{y \in Y \mid h^i(\mathcal{F} \otimes P_y) > 0\}$$

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where $\mathcal{F}$ is a sheaf or, more generally, a complex on X and $P_y := P_{|X\times\{y\}}$ is the locally free sheaf parametrized by y. One can also consider the integral transform

$$\mathbf{R}\Phi_P : \mathbf{D}(X) \to \mathbf{D}(Y), \quad \mathbf{R}\Phi_P(\cdot) := \mathbf{R}p_{Y*}(p_X^*(\cdot)\otimes P)$$

and the loci $\mathrm{Supp}(R^i\Phi_P\mathcal{F})$ in Y. An important point to note is that, although $\mathrm{Supp}(R^i\Phi_P\mathcal{F})$ is only contained in $V^i_P(\mathcal{F})$, and in general not equal to it, the sequences $\{\dim V^i_P(\mathcal{F})\}_i$ and $\{\dim \mathrm{Supp}(R^i\Phi_P\mathcal{F})\}_i$ carry the same basic numerical information, in the sense that for any integer k *the following conditions are equivalent*:

(a) $\mathrm{codim}_Y V^i_P(\mathcal{F}) \geq i-k$ for all $i \geq 0$.
(b) $\mathrm{codim}_Y\ \mathrm{Supp}(R^i\Phi_P\mathcal{F}) \geq i-k$ for all $i\geq 0$.

(This follows from a standard argument using base change, see Lemma 3.6 below.) If these conditions are satisfied, then $\mathcal{F}$ is said to *satisfy Generic Vanishing with index* $-k$ with respect to P or, for easy reference, to be a *GV_{-k}-sheaf* (or *GV_{-k}-object*).[1]

Note that if $\mathcal{F}$ is a GV_{-k}-sheaf, then the cohomological support loci $V^i_P(\mathcal{F})$ are proper subvarieties for $i > k$. When $k = 0$ we omit the index, and simply speak of GV-sheaves (or objects). An important example is given by the classical Green-Lazarsfeld theorem [GL1] which, in the above terminology, can be stated as follows: *let X be a smooth projective variety X, with Albanese map $a : X \to \mathrm{Alb}(X)$ and Poincaré line bundle P on $X \times \mathrm{Pic}^0(X)$. Then ω_X is $GV_{\dim a(X)-\dim X}$ with respect to P. In particular, if the Albanese map of X is generically finite, then ω_X is a GV-sheaf.*

The second point – sheaf vanishing – concerns the vanishing of the higher derived images $R^i\Phi_P\mathcal{G}$ for an object $\mathcal{G}$ in $\mathbf{D}(X)$. The question becomes interesting when the locus $V^i_P(\mathcal{G})$ is non-empty, since otherwise this vanishing happens automatically. Elementary base change tells us that, for a potential connection with the GV_{-k} condition above, the vanishing to look for has the following shape:

$$R^i\Phi_P\mathcal{G} = 0 \ \text{ for all } \ i < \dim X - k.$$

If this is the case, we say that *$\mathcal{G}$ is $WIT_{\geq(\dim X-k)}$ with respect to P*. The terminology is borrowed from Fourier-Mukai theory, where an object $\mathcal{G}$ on X is said *to satisfy the Weak Index Theorem (WIT) with index k* if $R^i\Phi_P\mathcal{G} = 0$ for $i \neq k$. Continuing with the example of the Poincaré bundle on $X \times \mathrm{Pic}^0(X)$, Hacon [Hac] proved that $R^i\Phi_P\mathcal{O}_X = 0$ for $i < \dim a(X)$ i.e. – in our terminology – *$\mathcal{O}_X$ is $WIT_{\geq \dim a(X)}$*. This was a conjecture of Green-Lazarsfeld (cf. [GL2] Problem 6.2; see also [Pa] for a different argument). In particular, if the Albanese map is generically finite, then $\mathbf{R}\Phi_P\mathcal{O}_X$ is a sheaf, concentrated in degree $\dim X$.

The third point expresses the vanishing of higher derived images in terms of the vanishing of a finite sequence of honest cohomology *groups*. A major step in this direction was made by Hacon for the case when the X is an abelian variety, Y its dual, and P is a Poincaré line bundle. (In this case, by Mukai's theorem [Muk], the functor $\mathbf{R}\Phi_P$ is an equivalence of categories.) This was a key point in his proof in [Hac] of the above mentioned conjecture of Green-Lazarsfeld.

An essential point of the present work is that Hacon's argument can be suitably refined so that it goes through for practically any integral transform $\mathbf{R}\Phi_P$, irrespective of whether it is an equivalence, or even fully faithful. To be precise, let us consider the analogous functor in the

[1]In a previous version of this paper, we used the term *GV_k-object* for what we now call a GV_{-k}-object. The reason for this change is that, as we will describe in upcoming work, the notion of Generic Vanishing index is useful and can be studied for an arbitrary integer k, making more logical sense with this sign convention.

opposite direction

$$\mathbf{R}\Psi_P : \mathbf{D}(Y) \to \mathbf{D}(X),\ \mathbf{R}\Psi_P(\cdot) := \mathbf{R}p_{X*}(p_Y^*(\cdot) \otimes P).$$

For a sufficiently positive ample line bundle A on Y, by Serre Vanishing we have $\mathbf{R}\Phi_P(A^{-1}) = R^g\Psi_P(A^{-1})[g]$. Moreover $R^g\Psi_P(A^{-1})$ is locally free, and we denote it $\widehat{A^{-1}}$. Following [Hac], one is lead to consider the cohomology groups $H^i(X, \mathcal{F} \otimes \widehat{A^{-1}})$. Denoting by $\mathbf{R}\Delta\mathcal{F} := \mathbf{R}\mathcal{H}om(\mathcal{F}, \omega_X)$ the (shifted) Grothendieck dual of $\mathcal{F}$, our general result is

Theorem A. *With the notation above, the following are equivalent:*

(1) $\mathcal{F}$ *is* GV_{-k}.

(2) $\mathbf{R}\Delta\mathcal{F}$ *is* $WIT_{\geq \dim X - k}$ *with respect to* $\mathbf{R}\Phi_{P^\vee} : \mathbf{D}(X) \to \mathbf{D}(Y)$.

(3) $H^i(X, \mathcal{F} \otimes \widehat{A^{-1}}) = 0$ *for any* $i > k$ *and any sufficiently positive ample line bundle* A *on* Y.

We refer to Theorem 3.7 for the most general hypotheses for Theorem A: X and Y do not necessarily need to be smooth, but rather just Cohen-Macaulay, and P does not have to be a locally free sheaf, but rather any perfect object in the bounded derived category of coherent sheaves. As already observed in [Hac], condition (3) in Theorem A is extremely useful when $Y = \mathrm{Pic}^0(X)$ and P is the Poincaré line bundle on $X \times \mathrm{Pic}^0(X)$, since in this case $\widehat{A^{-1}}$ has a very pleasant description: up to an étale cover of X, it it the direct sum of copies of the pullback of an ample line bundle via the Albanese map (cf. the proof of Theorem B below). This allows to reduce the Generic Vanishing conditions (1) or (2) to classical vanishing theorems.

The implication (2) $\Rightarrow$ (1) is a rather standard application of Grothendieck duality and of a basic fact on the support of Ext modules, well-known at least in the smooth case as a consequence of the Auslander-Buchsbaum formula (cf. Proposition 3.3). As mentioned above, the equivalence (2) $\Leftrightarrow$ (3) was proved in [Hac] for X and Y dual abelian varieties and P the Poincaré line bundle. Hence the novel points of Theorem A are the implication (1) $\Rightarrow$ (2) and the equivalence (2) $\Leftrightarrow$ (3) in the general setting of arbitrary integral transforms. The latter is already important in the well-studied case of the Poincaré line bundle P on $X \times \mathrm{Pic}^0(X)$, for the following technical reason. Deducing a result concerning the transform of $\mathcal{F}$ via $\mathbf{R}\Phi_P : \mathbf{D}(X) \to \mathbf{D}(\mathrm{Pic}^0(X))$ from a result involving the derived equivalence $\mathbf{R}\Phi_{\mathcal{P}} : \mathbf{D}(\mathrm{Alb}(X)) \to \mathbf{D}(\mathrm{Pic}^0(X))$, using the Albanese map $a : X \to \mathrm{Alb}(X)$, requires splitting and vanishing criteria for the object $\mathbf{R}a_*\mathcal{F}$ and its cohomologies. Such criteria are available for the canonical bundle in the form of Kollár's theorems on higher direct images of dualizing sheaves [Ko1], [Ko2]. These however require the use of Hodge theory, and are known not to hold in a more general setting, for example for line bundles of the type $\omega_X \otimes L$ with L nef.

We apply Theorem A to deduce such a Kodaira-type generalization of the Green-Lazarsfeld Generic Vanishing theorem to line bundles of the form $\omega_X \otimes L$ with L nef. When the nef part is trivial, one recovers the above results of Green-Lazarsfeld [GL1] and Hacon [Hac] – in this case, although the proof is just a variant of Hacon's, it has the following extra feature: Kollár's theorems on higher direct images of dualizing sheaves are not invoked, but rather just Kodaira-Kawamata-Viehweg vanishing which, according to Deligne-Illusie-Raynaud ([DI], [EV]), has an algebraic proof via reduction to positive characteristic. Thus we provide a purely algebraic proof of the Green-Lazarsfeld Generic Vanishing theorem, answering a question of Esnault-Viehweg ([EV], Remark 13.13(d)). To state the general theorem, we use the following notation: for a $\mathbb{Q}$-divisor L on X, we define κ_L to be $\kappa(L_{|F})$, the Iitaka dimension along the generic fiber of a, if $\kappa(L) \geq 0$, and 0 if $\kappa(L) = -\infty$.

Theorem B. *Let X be a smooth projective variety of dimension d and Albanese dimension $d-k$, and let P be a Poincaré line bundle on $X \times \mathrm{Pic}^0(X)$. Let L be a line bundle and D an effective $\mathbb{Q}$-divisor on X such that $L-D$ is nef. Then $\omega_X \otimes L \otimes \mathcal{J}(D)$ is a $GV_{-(k-\kappa_{L-D})}$-sheaf (with respect to P), where $\mathcal{J}(D)$ is the multiplier ideal sheaf associated to D. In particular, if L is a nef line bundle, then $\omega_X \otimes L$ is a $GV_{-(k-\kappa_L)}$-sheaf.*

Corollary C. *Let X be a smooth projective variety, and L a nef line bundle on X. Assume that either one of the following holds:*

(1) X is of maximal Albanese dimension.
(2) $\kappa(L) \geq 0$ and $L_{|F}$ is big, where F is the generic fiber of a.

Then $H^i(\omega_X \otimes L \otimes P) = 0$ for all $i > 0$ and $P \in \mathrm{Pic}^0(X)$ general. In particular $\chi(\omega_X \otimes L) \geq 0$, and if strict inequality holds, then $h^0(\omega_X \otimes L) > 0$. The same is true if we replace L by $L \otimes \mathcal{J}(D)$, where D is an effective $\mathbb{Q}$-divisor on X and L is a line bundle such that $L-D$ is nef.

Corollary D. *If X is a minimal smooth projective variety, then $\omega_X^{\otimes m}$ is $GV_{-(k-\kappa_F)}$, where κ_F is equal to the Kodaira dimension $\kappa(F)$ of the generic fiber of a if $\kappa(F) \geq 0$, and 0 otherwise.*

Note that in the above results, a high Iitaka dimension along the fibers compensates for the higher relative dimension of the Albanese map. In the same vein, we obtain a number of new generic vanishing results corresponding to standard vanishing theorems:

• generic Kollár-type vanishing criterion for higher direct image sheaves of the form $R^i f_* \omega_Y \otimes L$ with L nef (Theorem 5.8).
• generic Nakano-type vanishing criterion for bundles of holomorphic forms (Theorem 5.11).
• generic Le Potier, Griffiths and Sommese-type vanishing for vector bundles (Theorem 5.13).

In §6 we give some first applications of these results to birational geometry and linear series type questions, in the spirit of [Ko2] and [EL]. For instance, we show:

• the multiplicativity of generic plurigenera under étale maps of varieties whose Albanese map has generic fiber of general type (Theorem 6.2).
• the existence of sections, up to numerical equivalence, for weak adjoint bundles on varieties of maximal Albanese dimension (Theorem 6.1).

Here we exemplify only with a generalization of [EL] Theorem 3.

Theorem E. *Let X be a smooth projective variety such that $V^0(\omega_X)$ is a non-empty proper subset of $\mathrm{Pic}^0(X)$. Then $a(X)$ is ruled by positive-dimensional subtori of A.*[2]

We make some steps in the direction of higher rank Generic Vanishing as well. We give an example of a generic vanishing result for certain types of moduli spaces of sheaves on threefolds which are Calabi-Yau fiber spaces (cf. Proposition 7.7 for the slightly technical statement). The general such moduli spaces are pretty much the only moduli spaces of sheaves on varieties of dimension higher than two which seem to be quite well understood (due to work of Bridgeland and Maciocia [BM], based also on work of Mukai). A highly interesting point to be further understood here concerns the description and the properties, in relevant examples, of the vector bundles $\widehat{A^{-1}}$ used above. We also apply similar techniques to moduli spaces of bundles on a smooth projective curve to give a condition for a vector bundle to be a base point of any generalized theta linear series, extending a criterion of Hein [He] (cf. Corollary 7.5).

[2]Note that in case X is of maximal Albanese dimension, by generic vanishing the condition that $V^0(\omega_X)$ is a proper subset of $\mathrm{Pic}^0(X)$ is equivalent to $\chi(\omega_X) = 0$.

Finally, we mention that the results of this work have found further applications of a different nature in the context of abelian varieties in [PP3] (which establishes a connection with M-regularity) and [PP4] (which uses Theorem A to study subvarieties representing minimal cohomology classes in a principally polarized abelian variety).[3]

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2. Fourier-Mukai preliminaries

We work over a field k (in the applications essentially always over $\mathbb{C}$). Given a variety X, we denote by $\mathbf{D}(X)$ bounded derived category of coherent sheaves on X.

Let X and Y be projective varieties over k, and P a perfect object in $\mathbf{D}(X \times Y)$ (i.e. represented by a bounded complex of locally free sheaves of finite rank). This gives as usual two Fourier-Mukai-type functors

$$\mathbf{R}\Phi_P : \mathbf{D}(X) \to \mathbf{D}(Y), \ \mathbf{R}\Phi_P(\cdot) := \mathbf{R}p_{Y*}(p_X^*(\cdot)\underline{\otimes}P),$$

$$\mathbf{R}\Psi_P : \mathbf{D}(Y) \to \mathbf{D}(X), \ \mathbf{R}\Psi_P(\cdot) := \mathbf{R}p_{X*}(p_Y^*(\cdot)\underline{\otimes}P).$$

Projection formula and Leray isomorphism. We will use the same notation $H^i(\mathcal{E})$ for the cohomology of a sheaf and the hypercohomology of an object in the derived category. We recall the following standard consequence of the projection formula and the Leray isomorphism.

Lemma 2.1. *For all objects $\mathcal{E} \in \mathbf{D}(X)$ and $\mathcal{F} \in \mathbf{D}(Y)$,*

$$H^i(X, \mathcal{E}\underline{\otimes}\mathbf{R}\Psi_P\mathcal{F}) \cong H^i(Y, \mathbf{R}\Phi_P\mathcal{E}\underline{\otimes}\mathcal{F}).$$

Proof. By the projection formula (first and last isomorphism), and the Leray equivalence (second and third isomorphism) we have:

$$\begin{aligned} \mathbf{R}\Gamma(X, \mathcal{E}\underline{\otimes}\mathbf{R}\Psi_P\mathcal{F}) &\cong \mathbf{R}\Gamma(X, \mathbf{R}p_{X*}(p_X^*\mathcal{E}\underline{\otimes}p_Y^*\mathcal{F}\underline{\otimes}P)) \\ &\cong \mathbf{R}\Gamma(X \times Y, p_X^*\mathcal{E}\underline{\otimes}p_Y^*\mathcal{F}\underline{\otimes}P) \\ &\cong \mathbf{R}\Gamma(Y, \mathbf{R}p_{Y*}(p_X^*\mathcal{E}\underline{\otimes}p_Y^*\mathcal{F}\underline{\otimes}P)) \\ &\cong \mathbf{R}\Gamma(Y, \mathbf{R}\Phi_P\mathcal{E}\underline{\otimes}\mathcal{F}). \end{aligned}$$

□

Note that the statement works without any assumptions on the singularities of X and Y. We will use it instead of the more common comparison of Ext groups for adjoint functors of Fourier-Mukai functors on smooth varieties (cf. e.g. [BO] Lemma 1.2).

[3]For the sake of self-containedness, an ad-hoc proof of Theorem A in the case of the Fourier-Mukai transform between dual abelian varieties was given in [PP4], much simplified by the fact that in this case one deals with an equivalence of categories.

Grothendieck duality. We will frequently need to apply Grothendieck duality to Fourier-Mukai functors. Given a variety Z, for any object $\mathcal{E}$ in $\mathbf{D}(Z)$ the derived dual of $\mathcal{E}$ is

$$\mathcal{E}^\vee := \mathbf{R}\mathcal{H}om(\mathcal{E}, \mathcal{O}_Z).$$

When Z is Cohen-Macaulay and n-dimensional , we will also use the notation

$$\mathbf{R}\Delta\mathcal{E} := \mathbf{R}\mathcal{H}om(\mathcal{E}, \omega_Z),$$

so that the Grothendieck dualizing functor applied to $\mathcal{E}$ is $\mathbf{R}\Delta\mathcal{E}[n] = \mathbf{R}\mathcal{H}om(\mathcal{E}, \omega_Z[n])$.

Assume that X and Y are as above, with X Cohen-Macaulay, and P is an object in $\mathbf{D}(X \times Y)$. We use the notation introduced above.

Lemma 2.2. *The Fourier-Mukai and duality functors satisfy the following exchange formula:*

$$(\mathbf{R}\Phi_P)^\vee \cong \mathbf{R}\Phi_{P^\vee} \circ \mathbf{R}\Delta_X[\dim X].$$

Proof. We have the following sequence of equivalences:

$$\begin{aligned}
(\mathbf{R}\Phi_P(\cdot))^\vee &\cong \mathbf{R}\mathcal{H}om(\mathbf{R}p_{Y*}(p_X^*(\cdot)\underline{\otimes}P), \mathcal{O}_Y) \\
&\cong \mathbf{R}p_{Y*}(\mathbf{R}\mathcal{H}om(p_X^*(\cdot)\underline{\otimes}P, p_X^*\omega_X[\dim X])) \\
&\cong \mathbf{R}p_{Y*}(\mathbf{R}\mathcal{H}om(p_X^*(\cdot), p_X^*\omega_X[\dim X])\underline{\otimes}P^\vee) \\
&\cong \mathbf{R}p_{Y*}(p_X^*\mathbf{R}\Delta_X[\dim X](\cdot)\underline{\otimes}P^\vee) \\
&\cong \mathbf{R}\Phi_{P^\vee}(\mathbf{R}\Delta_X(\cdot)[\dim X]).
\end{aligned}$$

Besides basic operations allowed by the fact that we work with X projective[4], the main point is Grothendieck Duality in the second isomorphism. It works precisely as in the case of smooth morphisms, given that in this case the relative dualizing sheaf for p_Y is $p_X^*\omega_X$ (cf. [Ha] Ch.VII, §4). $\square$

Generalized WIT objects. A key concept in Fourier-Mukai theory is that of an object satisfying the Weak Index Theorem, generalizing terminology introduced by Mukai [Muk] in the context of abelian varieties. We consider again X and Y projective varieties over k, and P an object in $\mathbf{D}(X \times Y)$.

Definition/Notation 2.3. (1) An object $\mathcal{F}$ in $\mathbf{D}(X)$ is said to *satisfy the Weak Index Theorem with index j (WIT_j for short), with respect to P*, if $R^i\Phi_P\mathcal{F} = 0$ for $i \neq j$. In this case we denote $\widehat{\mathcal{F}} = R^j\Phi_P\mathcal{F}$.

(2) More generally, we will say that $\mathcal{F}$ satisfies $WIT_{\geq b}$ (or $WIT_{[a,b]}$ respectively) with respect to P, if $R^i\Phi_P\mathcal{F} = 0$ for $i < b$ (or for $i \notin [a, b]$ respectively).

Remark 2.4 ("Sufficiently positive")**.** In this paper we repeatedly use the notion of "sufficiently positive" ample line bundle on Y to mean, given any ample line bundle L, a power $L^{\otimes m}$ with $m \gg 0$. More precisely, for a finite collection of coherent sheaves on Y, m is sufficiently large so that by tensoring $L^{\otimes m}$ kills the higher cohomology of every sheaf in this collection.

We have a basic cohomological criterion for detecting WIT-type properties, as a consequence of Lemma 2.1 and Serre vanishing.

Lemma 2.5. *Let $\mathcal{F}$ be an object in $\mathbf{D}(X)$. Then*

$$R^i\Phi_P\mathcal{F} = 0 \iff H^i(X, \mathcal{F}\underline{\otimes}\mathbf{R}\Psi_P(A)) = 0$$

for any sufficiently positive ample line bundle A on Y.

[4] What one needs is that the resolution property for coherent sheaves be satisfied; cf. [Ha] Ch.II, §5.

Proof. We use the spectral sequence

$$E_2^{ij} := H^i(R^j\Phi_P\mathcal{F}\otimes A) \Rightarrow H^{i+j}(\mathbf{R}\Phi_P\mathcal{F}\otimes A).$$

By Serre vanishing, if A is sufficiently ample, then $H^i(R^j\Phi_P\mathcal{F}\otimes A) = 0$ for $i > 0$. Therefore $H^j(\mathbf{R}\Phi_P\mathcal{F}\otimes A) \cong H^0(R^j\Phi_P\mathcal{F}\otimes A)$ for all j. Hence $R^j\Phi_P\mathcal{F}$ vanishes if and only if $H^j(\mathbf{R}\Phi_P\mathcal{F}\otimes A)$ does. But, by Lemma 2.1, $H^j(\mathbf{R}\Phi_P\mathcal{F}\otimes A) \cong H^j(\mathcal{F}\underline{\otimes}\mathbf{R}\Psi_P A)$. □

Corollary 2.6. *Under the assumptions above, $\mathcal{F}$ satisfies WIT_j with respect to P if and only if, for a sufficiently positive ample line bundle A on Y,*

$$H^i(X, \mathcal{F}\underline{\otimes}\mathbf{R}\Psi_P(A)) = 0 \text{ for all } i \neq j.$$

More generally, $\mathcal{F}$ satisfies $WIT_{\geq b}$ (or $WIT_{[a,b]}$ respectively) with respect to P if and only if, for a sufficiently positive ample line bundle A on Y,

$$H^i(\mathcal{F}\underline{\otimes}\mathbf{R}\Psi_P(A)) = 0 \text{ for all } i < b \text{ (or } i \notin [a,b] \text{ respectively)}.$$

Example 2.7 (The graph of a morphism and Grauert-Riemenschneider)**.** The observation made in Lemma 2.5 is the generalization of a well-known Leray spectral sequence method used in vanishing of Grauert-Riemenschneider type. Let $f : X \to Y$ be a morphism of projective varieties, and consider $P := \mathcal{O}_\Gamma$ as a sheaf on $X\times Y$, where $\Gamma \subset X\times Y$ is the graph of f. Hence P induces the Fourier-Mukai functor $\mathbf{R}\Phi_P = \mathbf{R}f_*$, and $\mathbf{R}\Psi_P$ is the adjoint $\mathbf{L}f^*$.

The criterion of Lemma 2.5 then says that for an object $\mathcal{F}$ in $\mathbf{D}(X)$, we have

$$R^if_*\mathcal{F} = 0 \iff H^i(\mathcal{F}\otimes f^*A) = 0$$

for any A sufficiently ample on X. This is of course well known, and follows quickly from the Leray spectral sequence (cf. [La], Lemma 4.3.10). For instance, if X is smooth and f has generic fiber of dimension k, then $H^i(\omega_X\otimes L\otimes f^*A) = 0$ for all $i > k$ and all L nef on X, by an extension of Kawamata-Viehweg vanishing (cf. Lemma 5.1 and the comments before). This says that

$$R^if_*(\omega_X\otimes L) = 0, \text{ for all } i > k,$$

which is a more general (but with identical proof) version of Grauert-Riemenschneider vanishing.

Example 2.8 (Abelian varieties)**.** The statement of Lemma 2.5 is already of interest even in the case of an abelian variety X with respect to the classical Fourier-Mukai functors $\mathbf{R}\hat{\mathcal{S}} : \mathbf{D}(X) \to \mathbf{D}(\widehat{X})$ and $\mathbf{R}\mathcal{S} : \mathbf{D}(\widehat{X}) \to \mathbf{D}(X)$ (in Mukai's notation [Muk]) given by a Poincaré line bundle $\mathcal{P}$, i.e. in the present notation $\mathbf{R}\Phi_{\mathcal{P}}$ and $\mathbf{R}\Psi_{\mathcal{P}}$. Since A is sufficiently positive, $\mathbf{R}\mathcal{S}A = \widehat{A}$ is a vector bundle, and we simply have that $R^i\hat{\mathcal{S}}\mathcal{F} = 0$ iff $H^i(\mathcal{F}\otimes\widehat{A}) = 0$. Note that $\widehat{A}$ is negative, i.e. $\widehat{A}^\vee$ is ample (cf. [Muk] Proposition 3.11(1)).

3. GV-objects

In this section we introduce and study the notion of GV-object, which is modeled on many geometrically interesting situations. We will see that WIT and GV-objects are intimately related, essentially via duality. We use the notation of the previous section.

Definition 3.1. (1) An object $\mathcal{F}$ in $\mathbf{D}(X)$ is said to be a *GV-object with respect to P* if

$$\operatorname{codim}\operatorname{Supp}(R^i\Phi_P\mathcal{F}) \geq i \text{ for all } i \geq 0.$$

(2) More generally, for any integer $k \geq 0$, an object $\mathcal{F}$ in $\mathbf{D}(X)$ is called a *GV_{-k}-object with respect to P* if

$$\operatorname{codim}\operatorname{Supp}(R^i\Phi_P\mathcal{F}) \geq i-k \text{ for all } i \geq 0.$$

Thus $GV = GV_0$. (This definition is short hand for saying that $\mathcal{F}$ *satisfies Generic Vanishing with index k with respect to P.*)

(3) If $\mathcal{F}$ is a sheaf, i.e. a complex concentrated in degree zero, and the conditions above hold, we call $\mathcal{F}$ a *GV-sheaf*, or more generally a *GV_{-k}-sheaf.*

In the next three subsections we establish the main technical results of the paper, relating WIT and GV-objects. Together with Lemma 2.5, they will provide a cohomological criterion for checking these properties.

GV versus WIT. All throughout we assume that X and Y are Cohen-Macaulay. We set $d = \dim X$ and $g = \dim Y$.

Proposition 3.2 (GV implies WIT)**.** *Let $\mathcal{F}$ be an object in $\mathbf{D}(X)$ which is GV_{-k} with respect to P. Then $\mathbf{R}\Delta\mathcal{F}$ is $WIT_{\geq(d-k)}$ with respect to $P^\vee\underline{\otimes}p_Y^*\omega_Y$.*

Proof. Denote $Q := P^\vee\underline{\otimes}p_Y^*\omega_Y$. Let A be a sufficiently positive ample line bundle on Y. By Lemma 2.5, it is enough to prove that

$$H^i(X, \mathbf{R}\Delta\mathcal{F}\underline{\otimes}\mathbf{R}\Psi_Q A) = 0 \text{ for all } i < d-k.$$

By Grothendieck-Serre duality, this is equivalent to

$$H^j(X, \mathcal{F}\underline{\otimes}(\mathbf{R}\Psi_Q A)^\vee) = 0 \text{ for all } j > k.$$

By Lemma 2.2 we have $(\mathbf{R}\Psi_Q A)^\vee \cong \mathbf{R}\Psi_{Q^\vee}(\mathbf{R}\Delta A)[g] \cong \mathbf{R}\Psi_P(A^{-1})[g]$, where the second isomorphism follows by a simple calculation. In other words, what we need to show is

$$H^{j+g}(X, \mathcal{F}\underline{\otimes}\mathbf{R}\Psi_P(A^{-1})) = 0 \text{ for all } j > k.$$

This in turn is equivalent by Lemma 2.1 to

$$H^l(Y, \mathbf{R}\Phi_P\mathcal{F}\otimes A^{-1}) = 0 \text{ for all } l > g+k.$$

Now on Y we have a spectral sequence

$$E_2^{pq} := H^p(R^q\Phi_P\mathcal{F}\otimes A^{-1}) \Rightarrow H^{p+q}(\mathbf{R}\Phi_P\mathcal{F}\otimes A^{-1}).$$

Since $\mathcal{F}$ is GV_{-k} with respect to P, whenever $l = p+q > g+k$ we have that $\dim \mathrm{Supp}(R^q\Phi_P\mathcal{F}) < p$, and therefore $E_2^{pq} = 0$. This implies precisely what we want. $\square$

Proposition 3.3 (WIT implies GV)**.** *Let $\mathcal{F}$ be an object in $\mathbf{D}(X)$ which satisfies $WIT_{\geq(d-k)}$ with respect to $P\underline{\otimes}p_Y^*\omega_Y$. Then $\mathbf{R}\Delta\mathcal{F}$ is GV_{-k} with respect to $P^\vee$.*

Proof. Grothendieck duality (Lemma 2.2) gives

$$\mathbf{R}\Phi_{P^\vee}\big(\mathbf{R}\Delta\mathcal{F}\big) \cong (\mathbf{R}\Phi_P\mathcal{F}[d])^\vee \cong \mathbf{R}\Delta(\mathbf{R}\Phi_{P\underline{\otimes}p_Y^*\omega_Y}\mathcal{F}[d]).$$

By assumption, $\mathbf{R}\Phi_{P\underline{\otimes}p_Y^*\omega_Y}\mathcal{F}[d]$ is an object whose cohomologies R^j are supported in degrees at least $-k$. The spectral sequence associated to the composition of two functors implies in this case that there exists a spectral sequence

$$E_2^{ij} := \mathcal{E}xt^{i+j}(R^j, \omega_Y) \Rightarrow R^i\Phi_{P^\vee}(\mathbf{R}\Delta\mathcal{F}).$$

One can now use general facts on the support of Ext-sheaves. More precisely, since Y is Cohen-Macaulay, we know that

$$\mathrm{codim}\ \mathrm{Supp}(\mathcal{E}xt^{i+j}(R^j, \omega_Y)) \geq i+j$$

for all i and j. (This is better known as an application of the Auslander-Buchsbaum formula when the R^j have finite projective dimension, but holds in general by e.g. [BH] Corollary 3.5.11(c).) Since the only non-zero R^j-sheaves are for $j \geq -k$, we have that the codimension of the support of every E_∞ term is at least $i-k$. This implies immediately that codim $\mathrm{Supp}(R^i\Phi_{P^\vee}(\mathbf{R}\Delta\mathcal{F})) \geq i-k$, for all $i \geq 0$. □

Corollary 3.4. *Let $\mathcal{F}$ be an object in $\mathbf{D}(X)$ such that*

$$H^i(\mathcal{F}\underline{\otimes}\mathbf{R}\Psi_{P[g]}(A^{-1})) = 0 \text{ for all } i > k,$$

for any sufficiently positive ample line bundle A on Y. Then $\mathcal{F}$ is GV_{-k} with respect to P.

Proof. By Proposition 3.3, it is enough to prove that $\mathbf{R}\Delta\mathcal{F}$ is $WIT_{\geq(d-k)}$ with respect to $Q := P^\vee\underline{\otimes}p_Y^*\omega_Y$. Using Corollary 2.6 and Grothendieck-Serre duality, this follows as soon as

$$H^i(X, \mathcal{F}\otimes(\mathbf{R}\Psi_Q A)^\vee) = 0 \text{ for all } i > k.$$

We are left with noting that, by Lemma 2.2,

$$(\mathbf{R}\Psi_Q A)^\vee \cong \mathbf{R}\Psi_{P\underline{\otimes}p_Y^*\omega_Y^\vee[g]}(A^{-1}\otimes\omega_Y) \cong \mathbf{R}\Psi_{P[g]}(A^{-1}),$$

the last isomorphism being due to the Projection Formula and the fact that $\omega_Y\underline{\otimes}\omega_Y^\vee \cong \mathcal{O}_Y$. □

Cohomological support loci. Generic Vanishing conditions were originally given in terms of cohomological support loci, so it is natural to compare the definition of GV_{-k}-sheaves with the condition that the i-th cohomological support locus of $\mathcal{F}$ has codimension $\geq i-k$. For any $y \in Y$ we denote $P_y = \mathbf{L}i_y^*P$ the object in $\mathbf{D}(X)$, where $i_y : X\times\{y\} \hookrightarrow X\times Y$ is the inclusion.

Definition 3.5. Given an object $\mathcal{F}$ in $\mathbf{D}(X)$, the i-th cohomological support locus of $\mathcal{F}$ with respect to P is

$$V_P^i(\mathcal{F}) := \{y\in Y \mid \dim H^i(X, \mathcal{F}\underline{\otimes}P_y) > 0\}.$$

Lemma 3.6. *The following conditions are equivalent:*
(1) $\mathcal{F}$ is a GV_{-k}-object with respect to P.
(2) $\mathrm{codim}_Y V_P^i(\mathcal{F}) \geq i-k$ for all i.

Proof. [5] The statement follows from the theorem on cohomology and base change and its consequences.[6] Since by cohomology and base change $\mathrm{Supp}(R^i\Phi_P\mathcal{F}) \subseteq V_P^i(\mathcal{F})$, it is enough to prove that (1) implies (2). The proof is by descending induction on i. There certainly exists an integer s, so that $H^j(\mathcal{F}\underline{\otimes}P_y) = 0$ for any $j > s$ and for any $y \in Y$. Then, by base change, $\mathrm{Supp}(R^s\Phi_P\mathcal{F}) = V_P^s(\mathcal{F})$. The induction step is as follows: assume that there is a component $\bar{V}$ of $V_P^i(\mathcal{F})$ of codimension less than $i-k$. Since (1) holds, the generic point of $\bar{V}$ cannot be contained in $\mathrm{Supp}(R^i\Phi_P\mathcal{F})$ and so, again by base change, we have that $\bar{V} \subset V_P^{i+1}(\mathcal{F})$. This implies that $\mathrm{codim}_Y V_P^{i+1}(\mathcal{F}) < i-k$, which contradicts the inductive hypothesis. □

Here we will only use this in the standard setting where P and $\mathcal{F}$ are sheaves, with P locally free. In this case P_y is just the restriction of P to $X\times\{y\}$ and $V_P^i(\mathcal{F})$ is simply the locus where the sheaf cohomology $H^i(\mathcal{F}\otimes P_y)$ is non-zero.

The main technical result. One can put together the sequence of results above in order to obtain the main technical result, implying Theorem A in the Introduction.

[5] Cf. also [Hac] Corollary 3.2 and [Pa] Corollary 2.

[6] We recall that, although most commonly stated for cohomology of coherent sheaves (see e.g. the main Theorem of [Mum], §5 p.46), base change – hence also its corollaries, as [Mum] Corollary 3, §5 – works more generally for hypercohomology of bounded complexes ([EGA III] 7.7, especially 7.7.4, and Remarque 7.7.12(ii)).

Theorem 3.7. *Let X and Y be projective Cohen-Macaulay varieties, of dimensions d and g respectively, and let P be a perfect object in $\mathbf{D}(X \times Y)$. Let $\mathcal{F}$ be an object in $\mathbf{D}(X)$. The following conditions are equivalent:*

(1) $\mathcal{F}$ is a GV_{-k}-object with respect to P.

(2) $H^i(\mathcal{F} \underline{\otimes} \mathbf{R}\Psi_{P[g]}(A^{-1})) = 0$ for $i > k$ and any sufficiently positive ample line bundle A on Y.

(3) $R^i\Phi_{P^\vee \underline{\otimes} p_Y^*\omega_Y}(\mathbf{R}\Delta\mathcal{F}) = 0$ for all $i < d-k$ (i.e. $\mathbf{R}\Delta\mathcal{F}$ satisfies $WIT_{\geq(d-k)}$ with respect to $P^\vee \underline{\otimes} p_Y^*\omega_Y$).

(4) $\mathrm{codim}_Y V^i_P(\mathcal{F}) \geq i - k$ for all i.

Proof. Everything follows by putting together Proposition 3.2, Proposition 3.3, Corollary 3.4 and Lemma 3.6. □

Remark 3.8. To make the transition to the statement of Theorem A in the Introduction, simply note that we have

$$\mathbf{R}\Phi_{P^\vee \underline{\otimes} p_Y^*\omega_Y}(\mathbf{R}\Delta\mathcal{F}) \cong \mathbf{R}\Phi_{P^\vee}(\mathbf{R}\Delta\mathcal{F}) \underline{\otimes} \omega_Y.$$

In case Y is Gorenstein, ω_Y is a line bundle so (3) above is equivalent to $\mathbf{R}\Delta\mathcal{F}$ being $WIT_{\geq(d-k)}$ with respect to $P^\vee$.

Example 3.9 (The graph of a morphism II)**.** As in Example 2.7, let $f : X \to Y$ be a morphism of projective varieties, and consider $P := \mathcal{O}_\Gamma$ as a sheaf on $X \times Y$, where $\Gamma \subset X \times Y$ is the graph of f. We have $\mathbf{R}\Phi_P = \mathbf{R}f_*$ and $\mathbf{R}\Psi_P = \mathbf{L}f^*$. Assuming that X and Y are Cohen-Macaulay, the interpretation of Theorem 3.7 in this case is that an object $\mathcal{F}$ in $\mathbf{D}(X)$ is GV_{-k} with respect to P if and only if $H^i(\mathcal{F} \otimes f^*(A^{-1})) = 0$ for all $i > g + k$ and any A sufficiently positive on Y. In other words, in analogy with Example 2.7 we have the following, presumably folklore, consequence:

Corollary 3.10. *If f has generic fiber of dimension k, then for any object $\mathcal{F}$ in $\mathbf{D}(X)$:*

$$\mathrm{codim}\ \mathrm{Supp}(R^if_*\mathcal{F}) \geq i - k \iff H^i(\mathcal{F} \otimes f^*(A^{-1})) = 0,\ \forall i > g + k.$$

For example $\mathrm{codim}\ \mathrm{Supp}(R^if_*\mathcal{O}_X) \geq i - k$, for all i.

In most instances in which Theorem 3.7 is applied, due to geometrically restrictive assumptions on $\mathcal{F}$ and P, it is also the case that $R^i\Phi_{P^\vee}(\mathbf{R}\Delta\mathcal{F}) = 0$ for $i > d$.

Corollary 3.11. *If in Theorem 3.7 we assume in addition that the kernel P and $\mathcal{F}$ are sheaves, with P locally free, then $\mathcal{F}$ being GV_{-k} with respect to P is equivalent to*

(2′) $H^i(\mathcal{F} \otimes \mathbf{R}\Psi_{P[g]}(A^{-1})) = 0$ for $i \notin [0,k]$, and for any sufficiently positive ample line bundle A on Y.

(3′) $R^i\Phi_{P^\vee \otimes p_Y^*\omega_Y}(\mathbf{R}\Delta\mathcal{F}) = 0$ for all $i \notin [d-k, d]$ (i.e. $\mathbf{R}\Delta\mathcal{F}$ satisfies $WIT_{[d-k,d]}$ with respect to $P^\vee \otimes p_Y^*\omega_Y$).

Proof. By Lemma 2.5, the only thing we need to note is that under the extra assumptions we have $R^i\Phi_{P^\vee \otimes p_Y^*\omega_Y}(\mathbf{R}\Delta\mathcal{F}) = 0$ for $i > d$. Indeed, since $\mathcal{F}$ is a sheaf and P is locally free, $V_P^{d-i}(\mathcal{F}) = V^i_{P^\vee}(\mathbf{R}\Delta\mathcal{F})$ (Serre duality) is empty for $i > d$. But $V^i_{P^\vee}(\mathbf{R}\Delta\mathcal{F}) = V^i_{P^\vee \otimes p_Y^*\omega_Y}(\mathbf{R}\Delta\mathcal{F})$, so the assertion follows by base change. □

For convenience, it is worth stating the result of Theorem 3.7 in the most important special case, namely the relationship between $GV = GV_0$ and WIT_d under these geometric assumptions.

Corollary 3.12. *Under the assumptions of Theorem 3.7, if P and $\mathcal{F}$ are sheaves, and P is locally free, the following conditions are equivalent:*

(1) $\mathcal{F}$ is GV with respect to P.

(2) $H^i(\mathcal{F} \otimes \mathbf{R}\Psi_{P[g]}(A^{-1})) = 0$ for $i \neq 0$, for a sufficiently positive ample line bundle A on Y.

(3) $\mathbf{R}\Delta\mathcal{F}$ satisfies WIT_d with respect to $P^\vee \otimes p_Y^*\omega_Y$.

(4) $\mathrm{codim}_Y V_P^i(\mathcal{F}) \geq i$ for all i.

Remark 3.13. We emphasize that, as it follows from the proof (of Proposition 3.3), if the equivalent conditions of Corollary 3.12 hold, then

$$R^i\Phi_P\mathcal{F} \cong \mathcal{E}xt^i(\widehat{\mathbf{R}\Delta\mathcal{F}}, \omega_Y),$$

where the Fourier-Mukai hat is taken with respect to $P^\vee$ (cf. Definition/Notation 2.3).

Another remark which is very useful in applications (see e.g. [EL]) is the following (cf. also [Hac] Corollary 3.2(1) and [Pa] Corollary 2):

Proposition 3.14. *Assume that P is locally free. Let $\mathcal{F}$ be a GV_{-k}-sheaf with respect to P. Then*

$$V_P^d(\mathcal{F}) \subseteq \ldots \subseteq V_P^{k+1}(\mathcal{F}) \subseteq V_P^k(\mathcal{F}).$$

Proof. By Grothendieck-Serre duality we have for all i and all $y \in Y$ that

$$H^i(\mathcal{F} \otimes P_y) \cong H^{d-i}(\mathbf{R}\Delta\mathcal{F} \otimes P_y^\vee)^\vee.$$

If this is 0, then by base change (cf. e.g. [Mum] §5, Corollary 2, and also the comments in the proof of Lemma 3.6) the natural homomorphism

$$R^{d-i-1}\Phi_{P^\vee}(\mathbf{R}\Delta\mathcal{F}) \otimes k(y) \longrightarrow H^{d-i-1}(\mathbf{R}\Delta\mathcal{F} \otimes P_y^\vee)$$

is an isomorphism. Since $\mathcal{F}$ is GV_{-k} with respect to P, Theorem 3.7 and the Projection Formula imply that $R^{d-i-1}\Phi_{P^\vee}(\mathbf{R}\Delta\mathcal{F})$ is 0 for $i \geq k$, so again by duality we get that $H^{i+1}(\mathcal{F} \otimes P_y) = 0$. □

The case when V^0 is a proper subvariety. Assume that $\mathbf{R}\Phi_P$ is a Fourier-Mukai functor between X and Y projective Cohen-Macaulay, with P a locally free sheaf. When $V^0(\mathcal{F})$ is a proper subvariety of Y, one has strong consequences, useful in geometric applications (cf. §7). This generalizes results of Ein-Lazarsfeld for Albanese maps (cf. [EL] §1 and §2).

Proposition 3.15. *Assume that $\mathcal{F}$ is a sheaf on X which is GV with respect to P, and let W be a component of $V^0(\mathcal{F})$ of codimension p. If W is either isolated or of maximal dimension among the components of $V^0(\mathcal{F})$, then $\dim X \geq p$, and W is also a component of $V^p(\mathcal{F})$.*

Proof. By Grothendieck-Serre duality, the sheaf $\mathcal{G} := \widehat{\mathbf{R}\Delta\mathcal{F}}$ on Y (as in Remark 3.13) has support $V^0(\mathcal{F})$. Denote $\tau := \mathcal{G}_{|W}$ and consider the exact sequence given by restriction to W:

$$0 \to \mathcal{H} \to \mathcal{G} \to \tau \to 0.$$

Assuming that W is of maximal dimension among the components of $V^0(\mathcal{F})$, the codimension of the support of $\mathcal{H}$ is at least p, which implies that $\mathcal{E}xt^i(\mathcal{H}, \omega_Y) = 0$, for all $i < p$. As a consequence, for $i \leq p$ we have an inclusion $\mathcal{E}xt^i(\tau, \omega_Y) \hookrightarrow \mathcal{E}xt^i(\mathcal{G}, \omega_Y)$. This last assertion is

obviously also true if W is an isolated component of $V^0(\mathcal{F})$. On the other hand, it is standard that the support of $\mathcal{E}xt^p(\tau,\omega_Y)$ is W.

By Remark 3.13 we know that $\mathcal{E}xt^i(\mathcal{G},\omega_Y)\cong R^i\Phi_P\mathcal{F}$, which is 0 for $i>d$. Combined with the above, this gives $p\le \dim X$. In addition $V^p(\mathcal{F})$ contains the support of $\mathcal{E}xt^p(\mathcal{G},\omega_Y)$, which must contain the support of $\mathcal{E}xt^p(\tau,\omega_Y)$, i.e W. But since $\mathcal{F}$ is GV, $V^p(\mathcal{F})$ has codimension at least p, so W must then be one of its components. □

Corollary 3.16. *Assume that $\mathcal{F}$ is a sheaf on X which is GV with respect to P, such that $V^0(\mathcal{F})$ has an isolated point. Then* $\dim X\ge \dim Y$.

4. Examples of GV-objects

We have seen some basic examples related to morphisms in 2.7 and 3.9. In what follows we present a few other, more interesting, examples of GV-objects. With the exception of (5), all are in the following context: X is a smooth projective variety of dimension d, with Albanese map $a: X\to A$, P is a Poincaré bundle on $X\times\widehat{A}$, and $\mathbf{R}\Phi_P:\mathbf{D}(X)\to\mathbf{D}(\widehat{A})$ is the induced Fourier-Mukai functor. The example in (5) is related to §8.

(1) **The Green-Lazarsfeld Generic Vanishing theorem.** The main theorem of [GL1] says in the present language that if a has generic fiber of dimension k, then ω_X is a GV_{-k}-sheaf with respect to P. Green and Lazarsfeld also conjectured that if a is generically finite ($k=0$), then $R^i\Phi_P\mathcal{O}_X=0$ unless $i=d$, i.e. that $\mathcal{O}_X$ satisfies WIT_d with respect to P. This, and actually that in the general case $\mathcal{O}_X$ satisfies $WIT_{[d-k,d]}$, was proved by Hacon [Hac] and Pareschi [Pa]. Theorem 3.7 and Theorem B imply that this is in fact equivalent to the theorem in [GL1].

(2) **Line bundles on curves.** Let $X=C$, a smooth projective curve of genus g, so that $A\cong\widehat{A}\cong J(C)$, and a is an Abel-Jacobi map. If L is a line bundle on C, then L is GV with respect to P if and only if $\deg(L)\ge g-1$. Dually, L satisfies WIT_1 if and only if $\deg(L)\le g-1$. Every line bundle on C is GV_{-1}.

(3) **Line bundles on symmetric products and a result of Polishchuk.** A natural generalization of the previous example is as follows. For $1\le d\le g-1$, let $X=C_d$ be the d-th symmetric product of C. Its Albanese variety is $J(C)$ and the Albanese map is an abelian sum mapping $a: C_d\to J(C)$. Let also $\pi_d: C^d\to C_d$ be the natural projection. To a line bundle L on C, one can attach canonically a line bundle $F_d(L)$ on C_d such that $\pi_d^*F_d(L)\cong L^{\boxtimes d}$.

We claim that $\omega_{C_d}\otimes(F_d(L))^{-1}$ is GV with respect to $P^\vee$ (so also P) if and only if $\deg L\le g-d$. In order to prove this, we recall some standard facts (cf. e.g. [Iz] Appendix 3.1). In the first place, if $\xi\in\mathrm{Pic}^0(C)$ and P_ξ is the corresponding line bundle on $J(C)$, then $F_d(L)\otimes a^*P_\xi\cong(L\otimes P_\xi)^{\boxtimes d}$. Moreover $\pi^*\omega_{C_d}\cong\omega_C^{\boxtimes d}(-\Delta)$, where Δ is the sum of the big diagonals in C^d. Therefore

$$\pi_d^*(\omega_{C_d}\otimes(F_d(L)\otimes a^*P_\xi)^{-1})\cong(\omega_C\otimes(L\otimes P_\xi)^{-1})^{\boxtimes d}(-\Delta).$$

The cohomology of $\omega_{C_d}\otimes(F_d(L)\otimes a^*P_\xi)^{-1}$ is the skew-symmetric part of the cohomology of $(\omega_C\otimes(L\otimes\xi)^{-1})^{\boxtimes d}$ with the respect to the action of the symmetric group, so

$$H^i(C_d,\omega_{C_d}\otimes(F_d(L)\otimes a^*P_\xi)^{-1})\cong S^i(H^1(C,\omega_C\otimes(L\otimes\xi)^{-1}))\otimes\Lambda^{d-i}(H^0(C,\omega_C\otimes(L\otimes\xi)^{-1})).$$

A simple calculation shows that, for any i:

$$\mathrm{codim} V^i_{P^\vee}(\omega_{C_d}\otimes(F_d(L))^{-1})\ge i \iff \deg L\le g-d.$$

Although we mainly focus here on the GV notion, Theorem 3.7 can also be used dually to check that an object satisfies WIT, which is sometimes harder to prove. For example, using (1) $\Rightarrow$ (3), the calculation above gives a quick proof of a result of Polishchuk which essentially says:

Corollary 4.1 (cf. [Po2] Theorem 0.2)**.** *The line bundle $F_d(L)$ satisfies WIT_d with respect to P if and only if $\deg L \leq g - d$.*

(4) **Ideal sheaves of subvarieties in ppav's.** Consider now $X = J(C)$, the Jacobian of a smooth projective curve of genus g, with principal polarization Θ. For each $d = 1, \ldots, g-1$, denote by W_d the variety of special divisors of degree d in $J(C)$. In [PP1] Proposition 4.4, we proved that the sheaves $\mathcal{O}_{W_d}(\Theta)$ are M-regular and $h^0(\mathcal{O}_{W_d}(\Theta) \otimes \alpha) = 1$ for $\alpha \in \mathrm{Pic}^0(X)$ general. Using the twists of the standard exact sequence

$$0 \longrightarrow \mathcal{I}_{W_d}(\Theta) \longrightarrow \mathcal{O}_{J(C)}(\Theta) \longrightarrow \mathcal{O}_{W_d}(\Theta) \longrightarrow 0$$

by $\alpha \in \mathrm{Pic}^0(X)$, and Lemma 3.6, we deduce easily that $\mathcal{I}_{W_d}(\Theta)$ are GV-sheaves on $J(C)$. In the paper [PP4] we use Theorem 3.7 to go in the opposite direction: we show, among other things, that if $\mathcal{I}_Y(\Theta)$ is a GV-sheaf then Y represents a *minimal* class. This, in combination with a conjecture of Beauville-Debarre-Ran, means that the existence of subvarieties Y in a ppav (A, Θ) such that $\mathcal{I}_Y(\Theta)$ is GV should always characterize Jacobians, with the exception of intermediate Jacobians of cubic threefolds.

(5) **Stable sheaves on surfaces.** Consider X to be a complex abelian or $K3$ surface. For a coherent sheaf E on X, the Mukai vector of E is

$$v(E) := \mathrm{rk}(E) + c_1(E) + (\chi(E) - \epsilon \cdot rk(E))[X] \in H^{ev}(X, \mathbb{Z}),$$

where ϵ is 0 if X is abelian and 1 if X is $K3$. Given a polarization H on X and a vector $v \in H^{ev}(X, \mathbb{Z})$, we can consider the moduli space $M_H(v)$ of sheaves E with $v(E) = v$, stable with respect to H. If the Mukai vector v is primitive and isotropic, and H is general, the moduli space is $M = M_H(v)$ is smooth, projective and fine, and it is in fact again an abelian or $K3$ surface (cf. e.g. [Yo1]). The universal object $\mathcal{E}$ on $X \times M$ induces an equivalence of derived categories $\mathbf{R}\Phi_{\mathcal{E}} : \mathbf{D}(X) \to \mathbf{D}(M)$. Yoshioka gives many examples of situations in which in our language the GV (or, dually, the WIT) property is satisfied by $\mathcal{O}_X$ ($\cong \omega_X$) with respect to $\mathcal{E}$. Here we give just a very brief sampling. Some of these results will be used in Proposition 7.7.

Consider for example (X, H) to be a polarized $K3$ surface such that $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$, with $H^2 = 2n$. Let $k > 0$ be an integer, and consider $v = k^2 n + kH + [X]$. This is a primitive isotropic Mukai vector. Assume in addition that kH is very ample. It is shown in [Yo3] Lemma 2.4 that under these assumptions WIT_2 holds for $\mathcal{O}_X$ with respect to the kernel $\mathcal{E}^\vee$.[7] By Corollary 3.12, this is equivalent to the fact that $\mathcal{O}_X$ is a GV-sheaf with respect to $\mathcal{E}$.

There are similar examples when (X, H) is a polarized abelian surface with $\mathrm{Pic}(X) = \mathbb{Z} \cdot H$. Write $H^2 = 2r_0 k$, with $(r_0, k) = 1$. Consider the Mukai vector $v_0 := r_0 + c_1(H) + k[X]$, which is primitive and isotropic, so $M = M_H(v_0)$ is again an abelian surface, and there exists a universal object $\mathcal{E}$ on $X \times M$. It is proved in [Yo2], Theorem 2.3 and the preceding remark, that $\mathcal{O}_X$ (among many other examples) satisfies WIT_2 with respect to $\mathcal{E}^\vee$, which as above means that $\mathcal{O}_X$ is GV with respect to $\mathcal{E}$.

If (X, H) is a polarized abelian surface, one can consider the behavior of individual stable bundles with respect to the usual Fourier-Mukai functor $\mathbf{R}\hat{\mathcal{S}} : \mathbf{D}(X) \to \mathbf{D}(\widehat{X})$ as well. Assume

[7] It is also shown in *loc. cit.* §2 that in fact $X \cong M$.

for example that $NS(X) = \mathbb{Z} \cdot H$, and consider the Mukai vector $v = r + c_1(L) + a[X]$, where $a > 0$. It is shown in [Yo2] Proposition 1.1 that if E is a stable bundle with respect to H, with Mukai vector v, then E satisfies WIT_2 with respect to the dual Poincaré bundle $\mathcal{P}^\vee$, or in other words that $E^\vee$ is a GV-sheaf with respect to $\mathbf{R}\hat{\mathcal{S}}$.

5. Generic vanishing theorems

In this section we give the main applications of the material in §3, namely Generic Vanishing theorems related to the Picard variety. The main statement is phrased in the context of multiplier ideal sheaves. It generalizes results in [GL1], [Hac], and [Pa], and it contains as a special case generic vanishing for adjoint bundles of the form $K_X + L$ with L nef. We also show that many other standard constructions and vanishing theorems produce GV-sheaves. A key technical point, already noted by Hacon [Hac], is the very special nature of the Fourier-Mukai transform of an ample line bundle on an abelian variety.

Let X be smooth d-dimensional projective variety over a field of characteristic 0. Let $a : X \to A := \mathrm{Alb}(X)$ be the Albanese map of X, and assume that the dimension of $a(X)$ is $d - k$ and the dimension of A is g. Consider $\widehat{A} \cong \mathrm{Pic}^0(X)$ and $\mathcal{P}$ a Poincaré line bundle on $A \times \widehat{A}$. Consider also the pull-back $P := (a \times \mathrm{id}_{\widehat{A}})^*(\mathcal{P})$ on $X \times \widehat{A}$. In this context one can define Mukai's Fourier functors for abelian varieties

$$\mathbf{R}\hat{\mathcal{S}} : \mathbf{D}(A) \to \mathbf{D}(\widehat{A}) \text{ and } \mathbf{R}\mathcal{S} : \mathbf{D}(\widehat{A}) \to \mathbf{D}(A)$$

given by the kernel $\mathcal{P}$ in both directions and, more importantly for our purpose,

$$\mathbf{R}\Phi_P : \mathbf{D}(X) \to \mathbf{D}(\widehat{A}) \text{ and } \mathbf{R}\Psi_P : \mathbf{D}(\widehat{A}) \to \mathbf{D}(X)$$

as in §2. Every GV-type condition in this section will be with respect to P. Recall that for a nef $\mathbb{Q}$-divisor L on X, we denote by κ_L the Iitaka dimension $\kappa(L_{|F})$ of the restriction to the generic fiber, if $\kappa(L) \geq 0$, and 0 otherwise. We are now ready to prove Theorem B in the Introduction.

Proof. (*of Theorem B.*) *Step 1.* We first prove the Theorem in the case when D is an integral divisor, in other words we show the last assertion. Let L be a nef line bundle on X. It is enough to show that $\omega_X \otimes L$ satisfies condition (2) in Theorem 3.7 (cf. also Corollary 3.11).

Let M be ample line bundle on $\widehat{A}$, and assume for simplicity that it is symmetric, i.e. $(-1_{\widehat{A}})^*M \cong M$. We consider the two different Fourier transforms $\mathbf{R}\mathcal{S}M = R^0SM$ (on A) and $\mathbf{R}\Psi_{P[g]}(M^{-1}) = R^g\Psi_P(M^{-1}) =: \widehat{M^{-1}}$ (on X). These are both locally free sheaves. We first claim that

$$\widehat{M^{-1}} \cong a^*R^gSS(M^{-1}) \cong a^*(R^0\mathcal{S}M)^\vee.$$

The second isomorphism follows from Serre duality, the symmetry of M, and the fact that the Poincaré bundle satisfies the symmetry relation $\mathcal{P}^\vee \cong ((-1_A) \times 1_{\widehat{A}})^*\mathcal{P}$. If a were flat, which is usually not the case, the first isomorphism would follow simply by flat push-pull formula. However, the same result holds since both M and $R^gS(M^{-1})$ are locally free, hence flat. Less informally, consider the cartesian diagram

$$\begin{tikzcd} X \times \widehat{A} \arrow[r, "a \times id"] \arrow[d, "p_X"'] & A \times \widehat{A} \arrow[d, "p_A"] \\ X \arrow[r, "a"] & A \end{tikzcd}$$

Since p_A is flat, one can apply e.g. [Po1], Theorem on p. 276, saying that there is a natural isomorphism of functors

$$\mathbf{L}a^* \circ \mathbf{R}p_{A*} \cong \mathbf{R}p_{X*} \circ \mathbf{L}(a \times id_{\widehat{A}})^*.$$

The claim follows since, as we are dealing with locally free sheaves, the two derived pull-backs are the same as the usual pull-backs.[8]

On the other hand, by [Muk] 3.11, the vector bundle $R^0\mathcal{S}M$ has the property:

$$\phi_M^*(R^0\mathcal{S}M) \cong H^0(M) \otimes M^{-1}.$$

Here $\phi_M : \widehat{A} \to A$ is the standard isogeny induced by M. We consider then the fiber product $X' := X \times_A \widehat{A}$ induced by a and ϕ_M:

$$\begin{tikzcd} X' \arrow[r, "\psi"] \arrow[d, "b"'] & X \arrow[d, "a"] \\ \widehat{A} \arrow[r, "\phi_M"] & A \end{tikzcd}$$

It follows that

$$\psi^*\widehat{M^{-1}} \cong \psi^*a^*(R^0\mathcal{S}M)^\vee \cong b^*(H^0(M) \otimes M) \cong H^0(M) \otimes b^*M. \tag{1}$$

Recall that we want to prove the vanishing of $H^i(\omega_X \otimes L \otimes \widehat{M^{-1}})$ for $i > k - \kappa_L$. Since ψ, like ϕ_M, is étale, it is enough to prove this after pull-back to X', so for $H^i(X', \omega_{X'} \otimes \psi^*L \otimes \psi^*\widehat{M^{-1}})$. But by (1) we see that this amounts to the same for the groups $H^i(X', \omega_{X'} \otimes \psi^*L \otimes b^*M)$.

We next use the fact that in this particular setting we have additivity of Iitaka dimension in the following sense:

$$\kappa(\psi^*L \otimes b^*M) \geq d - k + \kappa_L.$$

In fact, assuming that $\kappa(L) \geq 0$, this is an actual equality since ψ^*L is nef and b^*M is the pull-back of an ample line bundle from a variety of dimension $d-k$, we have that

$$\kappa(\psi^*L \otimes b^*M) = \kappa(\psi^*L_{|F_b}) + d - k = \kappa(L_{|F}) + d - k,$$

where F_b is the generic fiber of b – cf. [Mo], Proposition 1.14.[9] On the other hand, if $\kappa(L) = -\infty$ it is still true that $\kappa(\psi^*L \otimes b^*M) \geq d - k$. (By cutting with general hyperplane sections we can reduce to the case of generically finite maps, when $\psi^*L \otimes b^*M$ is big and nef.)

Therefore the vanishing we want follows from the following well-known extension (and immediate consequence) of the Kawamata-Viehweg vanishing theorem for nef and big divisors (cf. [EV] 5.12):

$(*)$ *Let Z be a smooth projective variety and let N be a nef line bundle on Z, of Iitaka dimension $r \geq 0$. Then $H^i(Z, \omega_Z \otimes N) = 0$ for all $i > \dim Z - r$.*

Step 2. Consider now, in the general case, D an effective $\mathbb{Q}$-divisor on X and L a line bundle such that $L - D$ is nef. We keep the notation from the previous step. Let $f' : Y' \to X'$ be a log-resolution of the pair (X', ψ^*D). Since ψ is étale, by [La] 9.5.44 we have that $\mathcal{J}(\psi^*D) \cong \psi^{-1}\mathcal{J}(D) \cong \psi^*\mathcal{J}(D)$, and as in Step 1 we are reduced to showing the vanishing

$$H^i(X', \omega_{X'} \otimes b^*M \otimes \psi^*L \otimes \mathcal{J}(\psi^*D)) = 0, \text{ for all } i > k.$$

[8] Alternatively, in this particular case the claim would follow in fact for any coherent sheaf, since the map on the right is smooth – cf. [BO] Lemma 1.3.

[9] The set-up for the definitions and results in *loc.cit.* is for fiber spaces over normal varieties, but it is standard that the result can be easily reduced to that case by taking the Stein factorization.

The result follows from a Nadel-vanishing-type version of $(*)$, which is the content of the next Lemma.

Lemma 5.1. *Let X be a smooth projective complex variety of dimension d, D an effective $\mathbb{Q}$-divisor on X, and L a line bundle such that $L - D$ is nef and has Iitaka dimension $r \geq 0$. Then*

$$H^i(X, \omega_X \otimes L \otimes \mathcal{J}(D)) = 0, \text{ for all } i > d - r.$$

Proof. Like the proof of Nadel Vanishing, this is a standard reduction to the integral Kawamata-Viehweg-type statement, and we only sketch it. Take $f : Y \to X$ to be a log-resolution of the pair (X, D). By definition, $\mathcal{J}(D) = f_*\mathcal{O}_Y(K_{Y/X} - [f^*D])$. By Local Vanishing (cf. [La] Theorem 9.4.1) and the projection formula, using the Leray spectral sequence it is enough to have the desired vanishing for $H^i(Y, \mathcal{O}_Y(K_Y + f^*L - [f^*D]))$. As with the usual Kawamata-Viehweg theorem (cf. [La] 9.1.18), the statement is reduced via cyclic covers to the integral case, which is then covered by $(*)$. $\square$

This concludes the proof of the Theorem. $\square$

For applications it is useful to note that the statement of Theorem B can be extended to the setting of asymptotic multiplier ideals. In this case we do not need the nefness assumption. For a line bundle L with $\kappa(L) \geq 0$, we denote by $\mathcal{J}(\| L \|)$ the asymptotic multiplier ideal associated to L (cf. [La] §11.1).

Corollary 5.2. *Let X be a smooth projective variety of dimension d and Albanese dimension $d-k$, and L be a line bundle on X with $\kappa(L) \geq 0$. Then $\omega_X \otimes L \otimes \mathcal{J}(\| L \|)$ is a $GV_{-(k-\kappa_L)}$-sheaf.*

Proof. This is a corollary of the proof of Theorem B. By the behavior of asymptotic multiplier ideals under étale covers (cf. [La], Theorem 11.2.23), we have that $\psi^*\mathcal{J}(\| L \|) \cong \mathcal{J}(\| \psi^*L \|)$. As before, we need to show that for a sufficiently positive line bundle M on A we have

$$H^i(X', \omega_{X'} \otimes \psi^*L \otimes b^*M \otimes \mathcal{J}(\| \psi^*L \|)) = 0, \ \forall\ i > k - \kappa_L.$$

By definition we have $\mathcal{J}(\| \psi^*L \|) = \mathcal{J}(\frac{1}{p} \cdot |p\psi^*L|)$, where p is a sufficiently large integer. We consider $f : Y \to X'$ a log-resolution of the base locus of the linear series $|p\psi^*L|$, and write $f^*|p\psi^*L| = |M_p| + F_p$, where M_p is the moving part (in particular nef), and F_p is the fixed part. Thus $\mathcal{J}(\| \psi^*L \|) = f_*\mathcal{O}_Y(K_{Y/X'} - [\frac{1}{p}F_p])$, so by Local Vanishing and the projection formula, using the Leray spectral sequence we have

$$H^i(X', \omega_{X'} \otimes \psi^*L \otimes b^*M \otimes \mathcal{J}(\| \psi^*L \|)) \cong H^i(Y, \mathcal{O}_Y(K_Y + f^*\psi^*L - [\frac{1}{p}F_p] + f^*b^*M)).$$

Note now that $f^*\psi^*L - [\frac{1}{p}F_p]$ is numerically equivalent to the $\mathbb{Q}$-divisor $\frac{1}{p}M_p + \{\frac{1}{p}F_p\}$, where the last factor is fractional and simple normal crossings, so with trivial multiplier ideal. Since M_p is nef, as in Theorem B we have that $H^i = 0$ for $i > d - \kappa(M_p + f^*b^*M)$. On the other hand M_p asymptotically detects all the sections of ψ^*L, and the generic fiber for b and $b \circ f$ is the same, so we have that

$$\kappa(M_p + f^*b^*M) = d - k + \kappa_{M_p} = d - k + \kappa_L.$$

$\square$

Corollary 5.3. *Let X be a smooth projective variety whose Albanese map has generic fiber of general type (including the case of maximal Albanese dimension). Then $\mathcal{O}((m+1)K_X) \otimes \mathcal{J}(\| mK_X \|)$ is a GV-sheaf for all $m \geq 1$.*

Remark 5.4. (1) Theorem B holds more generally, but with the same proof, replacing the Albanese map with any morphism to an abelian variety $a : X \to A$.

(2) A particular case of Theorem B, extending the theorem of Green-Lazarsfeld to the case of line bundles of the form $\omega_X \otimes L$ with L semiample, or even nef and abundant, was proved by H. Dunio. This was done by reducing to the case of the Green-Lazarsfeld theorem via cyclic covers (cf. [EV] 13.7 and 13.10). Note also that Ch. Mourougane (cf. [Mou], Théoremè) showed using similar covering techniques that the same is true on a compact Kähler manifold. The main restriction in both cases essentially has to do, at least asymptotically, with the existence of lots of sections. We expect that Theorem B is also true on compact Kähler manifolds (X, ω) if L is only assumed to be nef, which in this case means that $c_1(L)$ is in the closure of the Kähler cone, i.e. the closed cone generated by smooth non-negative closed $(1, 1)$-forms.

(3) Results on the structure of cohomological support loci for twists with multiplier ideal sheaves, under some specific numerical hypotheses, are contained in [Bu].

Pluricanonical bundles. Generic Vanishing for ω_X states that if the Albanese dimension of X is $d - k$, then ω_X is a GV_{-k}-sheaf. Theorem B implies that the same is true for all powers $\omega_X^{\otimes m}$ when X is minimal. In fact, as soon as $m \geq 2$, one can do better according to the Kodaira dimension of the generic fiber of a. Recall that we denote by κ_F the Kodaira dimension $\kappa(F)$ if this is non-negative, or 0 if $\kappa(F) = -\infty$.

Corollary 5.5. *Let X be a smooth projective minimal variety. Then for any nef line bundle L on X and any $m \geq 1$, $\omega_X^m \otimes L$ is a $GV_{-(k-\kappa_F)}$-sheaf. In particular $\omega_X^{\otimes m}$ is a $GV_{-(k-\kappa_F)}$-sheaf for all $m \geq 2$.*

Proof. The minimality condition means that ω_X is nef. Everything follows directly from the proof Theorem B, together with the extra claim that $\kappa(F) = -\infty$ if and only if $\kappa(K_X + b^*M) = -\infty$. It is well known (cf. [Mo], Proposition 1.6), that $\kappa(F) = -\infty$ implies $\kappa(K_X + b^*M) = -\infty$. The statement that $\kappa(F) \geq 0$ implies $\kappa(K_X + b^*M) \geq 0$ follows from the general results on the subadditivity of Kodaira dimension (cf. *loc.cit.*). $\square$

As J. Kollár points out, it is very easy to see that if the minimality condition is dropped, then the higher powers $\omega_X^{\otimes m}$, $m \geq 2$, do not satisy generic vanishing.

Example 5.6. Let Y be the smooth projective minimal surface of maximal Albanese dimension (for example an abelian surface), and $f : X \to Y$ the blow-up of Y in a point, with exceptional divisor E. By the previous Corollary $\omega_Y^{\otimes m}$ is a GV-sheaf for all m. We claim that this cannot be true for $\omega_X^{\otimes m}$ when $m \geq 2$. Indeed, if one such were GV, then we would have that

$$\chi(\omega_X^{\otimes m}) = h^0(\omega_X^{\otimes m} \otimes f^*\alpha) \text{ and } \chi(\omega_Y^{\otimes m}) = h^0(\omega_Y^{\otimes m} \otimes \alpha)$$

for $\alpha \in \mathrm{Pic}^0(Y)$ general. The fact that $\omega_X \cong f^*\omega_Y \otimes \mathcal{O}_Y(E)$ implies easily that $h^0(\omega_X^{\otimes m} \otimes f^*\alpha) = h^0(\omega_Y^{\otimes m} \otimes \alpha)$. On the other hand, the Riemann-Roch formula shows that $\chi(\omega_X^{\otimes m}) \neq \chi(\omega_Y^{\otimes m})$ as soon as $m \geq 2$, which gives a contradiction.

Remark 5.7. The previous example shows that in general the tensor product of GV-sheaves is not GV. This is however true on abelian varieties, cf. [PP3].

Higher direct images. In analogy with work of Hacon (see [Hac] and also [HP] – cf. Remark 5.10 below), we show that higher direct images of dualizing sheaves have the same generic vanishing behavior as dualizing sheaves themselves. In order to have the most general statement, we replace the Albanese map of a smooth variety with the following setting: we consider X to

be an arbitrary Cohen-macaulay projective variety, and $a : X \to A$ a morphism to an abelian variety. We discuss the GV-property with respect to the Fourier-Mukai functor induced by the kernel $P = (a \times id)^*\mathcal{P}$ on $X \times \widehat{A}$, where $\mathcal{P}$ is the Poincaré bundle of $A \times \widehat{A}$.

Theorem 5.8. *Let $f : Y \to X$ be a morphism, with X, Y projective, Y smooth and X Cohen-Macaulay. Let L be a nef line bundle on $f(Y)$ (reduced image of f). If the dimension of $f(Y)$ is d and that of $a(f(Y))$ is $d-k$, then $R^j f_*\omega_Y \otimes L$ is a $GV_{-(k-\kappa_L)}$-sheaf on X for any j.*

Proof. This follows along the lines of the proof of Theorem B, so we only sketch the argument. Consider a sufficiently ample line bundle M on $\widehat{A}$, and form the two cartesian squares:

$$\begin{tikzcd} Y' \arrow[r, "f'"] \arrow[d, "\nu"] & X' \arrow[r, "b"] \arrow[d, "\psi"] & \widehat{A} \arrow[d, "\phi_M"] \\ Y \arrow[r, "f"] & X \arrow[r, "a"] & A \end{tikzcd}$$

where ϕ_M is the standard isogeny induced by M. As before, we need to check the vanishing

$$H^i(X', R^j f'_*\omega_{Y'} \otimes \psi^*L \otimes b^*M) = 0, \text{ for all } i > k \text{ and all } j.$$

We have that $\psi^*L \otimes b^*M$ is nef and has $\kappa(\psi^*L \otimes b^*M) \geq \kappa_L + d - k$. The required vanishing is a consequence of the variant of Kollár's vanishing theorem (cf. [Ko1], Theorem 2.1(iii)) stated in the Lemma below. (Note that the Lemma is applied replacing X' with $f'(Y')$). $\square$

Lemma 5.9. *Let $f : Y \to X$ be a surjective morphism of projective varieties, with Y smooth and X of dimension d. If M is a nef line bundle on X, of Kodaira-Iitaka dimension $\kappa(M) = d-k$, then*

$$H^i(X, R^j f_*\omega_Y \otimes M) = 0, \text{ for all } i > k \text{ and all } j.$$

Proof. This uses the full package provided by [Ko1] Theorem 2.1. We note to begin with that the condition on M implies that there exist hypersurfaces $H_1, \ldots, H_k$ on X such that $Z := H_1 \cap \ldots \cap H_k \subset X$ is a subvariety of dimension $d-k$ such that $M_{|Z}$ is big and nef. We can assume that the H_i's are sufficiently positive and general, so in particular by Bertini the preimages $\tilde{H}_i$ of the H_i's in Y are smooth.

We do a descending induction: let H in X be one of the hypersurfaces H_i as above. Pushing forward the obvious adjoint sequence on Y, we obtain a long exact sequence:

$$\ldots \to R^{j-1}f_*\omega_{\tilde{H}} \to R^j f_*\omega_Y \to R^j f_*(\omega_Y(\tilde{H})) \to R^j f_*\omega_{\tilde{H}} \to R^{j+1}f_*\omega_Y \to \ldots$$

The sheaves $R^j f_*\omega_{\tilde{H}}$ are supported on H, while by [Ko1] Theorem 2.1(i), the sheaves $R^j f_*\omega_Y$ are torsion-free. This implies that the long exact sequence above breaks in fact into short exact sequences

$$0 \to R^j f_*\omega_Y \to R^j f_*(\omega_Y(\tilde{H})) \to R^j f_*\omega_{\tilde{H}} \to 0.$$

We twist these exact sequences by M, and pass to cohomology. Since H can be taken sufficiently positive and $f_*\mathcal{O}_Y(\tilde{H}) \cong \mathcal{O}_X(H)$, by Serre vanishing we may assume that $H^i(R^j f_*(\omega_{\tilde{Y}}(\tilde{H})) \otimes M) = 0$ for all $i > 0$. This implies that $H^i(R^j f_*\omega_Y \otimes M) \cong H^{i-1}(R^j f_*\omega_{\tilde{H}} \otimes M)$ for all i and all j. We continue intersecting with H_i's until we get to Z. This implies that

$$H^i(X, R^j f_*\omega_Y \otimes M) \cong H^{i-k}(Z, R^j f_*\omega_{\tilde{Z}} \otimes M_{|Z}).$$

But this is 0 for $i > k$, since on Z we can apply the vanishing theorem [Ko1] Theorem 2.1(iii) (or rather its well-known version for big and nef line bundles). $\square$

Remark 5.10. In the case when one considers higher direct images $R^j f_* \omega_Y$, i.e. $L = \mathcal{O}_X$, Theorem 5.8 was already noted in [Hac] Corollary 4.2, and more generally [HP] Theorem 2.2(a) and the references therein.

Generic Nakano-type vanishing. Using similar techniques, we can deduce generic vanishing results for bundles of holomorphic forms, based on a suitable generalization of Nakano vanishing.

Theorem 5.11. *Let X be a smooth projective variety, with Albanese image of dimension $d-k$. Denote by m the* maximal *dimension of a fiber of a, and consider $l := \max\{k, m-1\}$. Then:*
(1) Ω_X^j is a $GV_{-(d-j+l)}$-sheaf for all j.
(2) $\operatorname{codim}_{\hat{A}} V^i(\Omega_X^j) \geq \max\{i+j-d-l, d-i-j-l\}$, for all i and all j.
(3) If L is a nef line bundle on X and a is finite*, then $\Omega_X^j \otimes L$ is a $GV_{-(d-j)}$-sheaf for all j.*

Proof. (1) Again we follow precisely the pattern of the proof of Theorem B, replacing $\omega_X \otimes L$ with Ω_X^j. The result reduces to checking the vanishing

$$H^i(X', \Omega_{X'}^j \otimes b^* M) = 0, \text{ for all } i > d-j+l.$$

But, as the pull-back of an ample line bundle via b, the line bundle b^*M is m-ample in the sense of Sommese ([EV] 6.5). Thus the needed vanishing follows from the generalization by Sommese and Esnault-Viehweg of the Nakano vanishing theorem: if L is an m-ample line bundle on Y of dimension d, then $H^i(Y, \Omega_Y^j \otimes L) = 0$ for all $i > d-j+\max\{d-\kappa(L), m-1\}$ (cf. [EV] 6.6).
(2) Apply part (1) to the sheaves Ω_X^j and Ω_X^{d-j}, which are related in an obvious way by Serre duality.
(3) As before, we are reduced to checking the vanishing of the cohomology groups $H^i(X', \Omega_{X'}^j \otimes \psi^* L \otimes b^* M)$. But since b is a finite map, $\psi^* L \otimes b^* M$ is ample, and so the result follows from the Nakano vanishing theorem. □

Remark 5.12. Note that Nakano vanishing does not hold in the more general setting of twisting with big and nef line bundles (cf. [La], Example 4.3.4). Thus one does not expect to have generic vanishing for Ω_X^j depending only on the dimension of the generic fiber of the Albanese map, as in the case of ω_X. A counterexample was indeed given by Green and Lazarsfeld ([GL1], Remark after Theorem 3.1). The above shows that there are however uniform bounds depending on the *maximal* dimension of the fibers of the Albanese map – in particular if the Albanese map is equidimensional, or if the fiber dimension jumps only by one, then the exact analogue of generic vanishing for ω_X does hold.[10] Note that in the counterexample mentioned above, the fiber dimension jumps by 2: it is shown there that Ω_X^1 does not satisfy the expected GV condition. The result above shows that it does satisfy the "one worse" GV condition. Finally, in [GL1] Theorem 2, another variant for generic Nakano-type vanishing is proposed in terms of zero-loci of holomorphic 1-forms. It would be interesting if the two approaches could be combined.

Vector bundles. By the same token, the various known vanishing theorems for higher rank vector bundles show that nef vector bundles also satisfy a weaker form of generic vanishing. Below is a sampling of results. All the notions and results we refer to can be found in [La] §7.3.

Theorem 5.13. *Let X be a smooth projective variety of dimension d and E a nef vector bundle on X. Then:*
(1) If the Albanese map of X is finite*, then $\omega_X \otimes \Lambda^a E$ is a $GV_{-(\mathrm{rk}(E)-a)}$-sheaf. Moreover*

[10]For example, in the case of finite Albanese maps, the results of [GL1] seem to imply only the weak form of generic vanishing, as in our Corollary C in the Introduction.

$\Omega_X^j \otimes E$ *is a* $GV_{-(\mathrm{rk}(E)+d-j-1)}$*-sheaf for all* j.
(2) If the Albanese map of X *is* generically finite*, then for all* $m \geq 0$, $\omega_X \otimes S^m E \otimes \det(E)$ *is a* GV*-sheaf. Moreover, if* E *is* k*-ample in the sense of Sommese, then* $\omega_X \otimes \Lambda^a E$ *is a* $GV_{-(\mathrm{rk}(E)+k-a)}$*-sheaf for all* $a > 0$.

Proof. Everything goes exactly as in the proof of Theorem B, so in the end one is reduced to checking vanishing for cohomology groups of the form $H^i(\omega_X \otimes F \otimes L)$ and the corresponding Akizuki-Nakano analogues, where F is a vector bundle as above, and L is an ample or big and nef line bundle. Then (1) follows from the Le Potier vanishing theorem, the second part of (2) follows from Sommese's version of the same theorem, while the first part of (2) from the Griffiths vanishing theorem (which also works in the big and nef case – cf. [La] 7.3.2 and 7.3.3). $\square$

Further, more refined results along these lines, and especially taking into account the beautiful general vanishing theorems for vector bundles of Demailly, Manivel, Arapura and others, can be formulated by the interested reader. In the Kähler case, results on Nakano semipositive vector bundles with some stronger conditions on the twists were proved by Mourougane [Mou].

Algebraicity and positive characteristic. The methods of this paper give an algebraic proof of Generic Vanishing in characteristic 0. The main point is that the Kawamata-Viehweg theorem can be reduced in characteristic 0 to Kodaira vanishing, via covering constructions. This in turn is proved via reduction mod p in [DI]. We will comment later that other known results can be similarly proved algebraically – cf. Remark 6.7.

On the other hand, assume that X is defined over a perfect field of characteristic $\mathrm{char}(k) \geq \dim(X)$, and that it admits a lifting to the 2nd Witt vectors $W_2(k)$. Then, again by [DI], Kodaira vanishing is still known to hold. However, this is not (yet) the case with the analogue of Kawamata-Viehweg (cf. [EV] 11.6 and 11.7) – it would follow as in characteristic 0 if we had embedded resolution of singularities over the field k and over $W_2(k)$.

As a consequence, with the current state of knowledge we know that the main results on Generic Vanishing in this paper (especially Theorem B) hold in positive characteristic, under the assumptions above, only if either one of the following holds:
(1) The Albanese map is finite onto its image.
(2) The dimension of X is at most three (cf. [Ab]), if $\mathrm{char}(k) > 5$ and the embedded resolution also admits a lifting over $W_2(k)$.
(3) The standard generic vanishing for ω_X holds in arbitrary characteristic if the Albanese map is separable, by a result of the first author [Pa].

6. Applications via the Albanese map

We give some examples of how Theorems B and 5.8 can be applied to basic questions in the spirit of [Ko2] and [EL], and we make another comment on algebraic proofs.

Existence of sections and generic plurigenera. The first is related to the existence of sections. Recall that it is known that every nef line bundle on an abelian variety is numerically equivalent to an effective one. Also, if L is a nef and big line bundle on a variety of maximal Albanese dimension, then $K_X + L$ has a non-zero section.[11]

Theorem 6.1. *Let* X *be a smooth projective variety of maximal Albanese dimension. Let* L *be a line bundle on* X *such that either one of the following holds:*

[11] Cf. e.g. [PP2] §5 for a quick proof of this; in fact much more holds, cf. [Ko1] Theorem 16.2.

(1) L is nef.
(2) $\kappa(L) \geq 0$.

Then there exists $\alpha \in \mathrm{Pic}^0(X)$ such that $h^0(\omega_X \otimes L \otimes \alpha) > 0$. In particular $K_X + L$ is numerically equivalent to an effective divisor.

Proof. Assume first that L is nef. If the conclusion doesn't hold, we have that $V^0(\omega_X \otimes L) = \emptyset$. But since by the Theorem $\omega_X \otimes L$ is GV, Proposition 3.14 implies that $V^i(\omega_X \otimes L) = \emptyset$ for all i. By Grauert-Riemenschneider vanishing we have that $R^j a_*(\omega_X \otimes L) = 0$ for all $j > 0$, so $V^i(a_*(\omega_X \otimes L)) = V^i(\omega_X \otimes L)$ for all i. This means that $a_*(\omega_X \otimes L)$ is a non-zero (since a is generically finite) sheaf on A whose Fourier-Mukai transform $\mathbf{R}\hat{\mathcal{S}}(a_*(\omega_X \otimes L))$ is equal to zero. But this is impossible since $\mathbf{R}\hat{\mathcal{S}}$ is an equivalance.

If L has non-negative Iitaka dimension, we can consider the asymptotic multiplier ideal $\mathcal{J}(\| L \|)$ as in Corollary 5.2, and we show that in fact there exists $\alpha \in \mathrm{Pic}^0(X)$ such that

$$h^0(\omega_X \otimes L \otimes \mathcal{J}(\| L \|) \otimes \alpha) > 0.$$

As above, by Corollary 5.2 $\omega_X \otimes L \otimes \mathcal{J}(\| L \|)$ is a GV-sheaf, so assuming that the conclusion doesn't hold, we get a contradiction if we know that $V^i(a_*(\omega_X \otimes L \otimes \mathcal{J}(\| L \|))) = V^i(\omega_X \otimes L \otimes \mathcal{J}(\| L \|))$ for all i, which in turn follows if we know that

$$R^j a_*(\omega_X \otimes L \otimes \mathcal{J}(\| L \|)) = 0 \text{ for all } j > 0.$$

Recall that $\mathcal{J}(\| L \|) = \mathcal{J}(\frac{1}{m} \cdot |mL|)$ for some $m \gg 0$, and consider $\phi : Y \to X$ a log-resolution of the base locus of $|mL|$. Write $\phi^*(mL) = M_m + F_m$, where M_m is the moving part and F_m the fixed part, in simple normal crossings. We see easily that

$$R^j a_*(\mathcal{O}_X(K_X + L) \otimes \mathcal{J}(\| L \|)) \cong R^j(a \circ \phi)_* \mathcal{O}_Y(K_Y + N),$$

where $N \equiv_{\mathbb{Q}} \frac{1}{m} M_m + \{\frac{1}{m} F_m\}$. Since M_m is nef and $a \circ \phi$ is generically finite, this follows again from the ($\mathbb{Q}$-version of) Grauert-Riemenschneider-type vanishing (cf. [KM] Corollary 2.68, noting that the same proof works for generically finite maps). $\square$

Next we use Corollary 5.3 for a result which interpolates between Kollár's theorem on the multiplicativity of plurigenera under étale maps for varieties of general type (cf. [Ko2] Theorem 15.4), and the (obvious) case of abelian varieties. The invariant which is well-behaved under étale covers in this case is the *generic plurigenus*:

$$P_{m,gen} := h^0(X, \mathcal{O}(mK_X) \otimes \alpha),$$

where $\alpha \in \mathrm{Pic}^0(X)$ is taken general enough so that the quantity is minimal over $\mathrm{Pic}^0(X)$.

Theorem 6.2. *Let $Y \to X$ be an étale map of degree e between smooth projective varieties whose Albanese maps have generic fiber of general type (including the case of maximal Albanese dimension). Then*

$$P_{m,gen}(Y) = e \cdot P_{m,gen}(X), \text{ for all } m \geq 2.$$

Proof. The proof follows Kollár idea of expressing the plurigenus as an Euler characteristic, but in the language of asymptotic multiplier ideals as in [La] Theorem 11.2.23. Fix $m \geq 2$ and $\alpha \in \mathrm{Pic}^0(X)$ sufficiently general so that

$$P_{m,gen}(X) = h^0(X, \mathcal{O}(mK_X) \otimes \alpha).$$

Since torsion points are dense in $\mathrm{Pic}^0(X)$, we can further assume that α is torsion. The point is that asymptotic multiplier ideals do not detect torsion. Indeed:

$$\mathcal{J}(\| mK_X + \alpha \|) \cong \mathcal{J}(\| mK_X \|),$$

since these are computed from the linear series $|p(mK_X+\alpha)|$ and $|pmK_X|$, for any p sufficiently large, and in particular divisible enough so that it kills α. On the other hand, we know that

$$H^0(\mathcal{O}(mK_X)\otimes\alpha) \cong H^0(\mathcal{O}(mK_X)\otimes\alpha\otimes\mathcal{J}(\| mK_X+\alpha \|)),$$

and also that $\mathcal{J}(\| mK_X \|) \subseteq \mathcal{J}(\| (m-1)K_X \|)$, so as a consequence we have

$$P_{m,gen}(X) = h^0(\mathcal{O}(mK_X)\otimes\alpha\otimes\mathcal{J}(\| (m-1)K_X \|)).$$

At this stage we can use Generic Vanishing: by Corollary 5.3, we know that the sheaf $\mathcal{O}(mK_X)\otimes \mathcal{J}(\| (m-1)K_X \|)$ is GV, so α could also be chosen such that

$$H^i(\mathcal{O}(mK_X)\otimes\alpha\otimes\mathcal{J}(\| (m-1)K_X \|)) = 0,\ \forall\ i>0.$$

This finally implies that

$$P_{m,gen}(X) = \chi(\mathcal{O}(mK_X)\otimes\mathcal{J}(\| (m-1)K_X \|)),$$

since Euler characteristic is invariant under deformation. Since the same is true for Y, the result follows immediately from the multiplicativity of Euler characteristics under étale maps, and the fact that

$$\mathcal{J}(\| (m-1)K_Y \|) \cong f^*\mathcal{J}(\| (m-1)K_X \|),$$

which is a consequence of the behavior of asymptotic multiplier ideals under étale covers, [La] Theorem 11.2.16. □

This implies subadditivity of generic plurigenera, precisely as in [Ko1] Theorem 15.6.

Corollary 6.3. *Let X be a smooth projective variety with nontrivial algebraic fundamental group, whose Albanese map has generic fiber of general type (for example a variety of maximal Albanese dimension). Then, for $m,n\geq 2$, if $P_{m,gen}(X)>0$ and $P_{n,gen}(X)>0$, then*

$$P_{m+n,gen}(X) \geq P_{m,gen}(X) + P_{n,gen}(X).$$

Since a variety X of maximal Albanese dimension has generically large fundamental group (see [Ko2] Definition 4.6 for the slightly technical definition), by [Ko2] Theorem 16.3 we have that if X is of general type, then $P_m(X)\geq 1$ for $m\geq 2$ and $P_m(X)\geq 2$ for $m\geq 4$. Using the above, a stronger statement can be made.

Corollary 6.4. *Let X be a variety of maximal Albanese dimension and of general type. Then*
(1) $P_{m,gen}(X)\geq 1$ for $m\geq 2$.
(2) $P_{m,gen}(X)\geq 2$ for $m\geq 4$.

Proof. The first statement is already known, in a more general form. Indeed, since bigness is preserved under numerical equivalence, we have that $\omega_X\otimes\alpha$ is big for all $\alpha\in\mathrm{Pic}^0(X)$. But on varieties of maximal Albanese dimension (or more generally with generically large fundamental group), every line bundle of the form $\omega_X\otimes L$ with L big has non-zero sections (cf. [Ko2] Theorem 16.2). The second follows from (1) and Corollary 6.3. □

Components of V^0. Let X be a smooth projective variety of dimension d, and $a: X\to A$ its Albanese map. We denote as always the dimension of the generic fiber of a by k. We can assume without loss of generality that the image $Y=a(X)$ is smooth by passing to a resolution

of singularities, which is sufficient to check the dimension properties we are interested in. First of all, Corollary 3.16 gives in this case:

Corollary 6.5. *Say X is of maximal Albanese dimension, and there exists a GV-sheaf $\mathcal{F}$ on X with an isolated point in $V^0(\mathcal{F})$. Then* $\dim X \geq \dim A$*, and so the Albanese map is surjective and* $\dim X = \dim A$.

A special case is the following extension of [EL] Proposition 2.2; cf. also Remark 6.7.

Corollary 6.6. *Assume that the characteristic is zero, and that there exists an isolated point in $V^0(\omega_X \otimes a^*L)$ for some nef line bundle L on $a(X)$. Then the Albanese map a is surjective.*

Proof. By passing to a resolution of singularities, we can assume that the Albanese image $Y = a(X)$ is smooth. Theorem 5.8 implies that $a_*\omega_X \otimes L$ is a GV-sheaf on Y. By hypothesis there is an isolated point in $V^0(a_*\omega_X \otimes L)$, so Corollary 6.5 applies to give $Y = A$. □

The main result is that Theorem 5.8 and Proposition 3.15, together with the structure result for $V^0(\omega_X)$ in [GL2], imply the generalization of [EL] Theorem 3 mentioned in the Introduction.

Proof. (*of Theorem E.*) Assume that there is a component W of $V^0(\omega_X)$ of codimension $p > 0$. By [GL2] we know that W is a translation of an abelian subvariety of $\widehat{A} = \mathrm{Pic}^0(X)$, which we will abusively also denote by W. Denote $C := \widehat{W}$, the dual abelian variety, and consider the sequence of homomorphisms of abelian varieties

$$1 \to B \to A \to C \to 1.$$

Let $Y = a(X)$, and consider the morphism $f : Y \to C$ induced by the composition of the inclusion in A and the projection to C. Denote by k the dimension of the generic fiber of f. To prove the Theorem it is enough to show that $k \geq p$. Indeed, the fibers of $A \to C$ are subtori of A of dimension p.

Now Theorem 5.8 implies that $a_*\omega_X$ is a GV-sheaf on $Y = a(X)$, and we clearly have $V^0(a_*\omega_X) = V^0(\omega_X)$. This, together with Proposition 3.15, implies that W is also a component of $V^p(a_*\omega_X)$. On the other hand, Theorem 5.8 also implies that $a_*\omega_X$ is a GV_{-k}-sheaf with respect to the natural Fourier-Mukai transform $\mathbf{D}(Y) \to \mathbf{D}(C)$, so that

$$\operatorname{codim} V^p_{\widehat{C}}(a_*\omega_X) \geq p - k.$$

But $\widehat{C} = W$, so by definition the line bundles in $\mathrm{Pic}^0(X)$ parametrized by W are precisely those pulled back from C. We finally have that $W \subseteq V^p_{\widehat{C}}(a_*\omega_X) \subseteq \widehat{C} = W$, which implies equality everywhere. Hence the codimension of $V^p_{\widehat{C}}(a_*\omega_X)$ is in fact 0, which gives $k \geq p$. □

Remark 6.7 (Algebraic proofs)**.** We note that in the present approach one does not need to appeal to complex analytic techniques. For instance, the Theorem above gives in particular an algebraic proof of [EL] Theorem 3. Moreover, the case $L = \mathcal{O}_X$ in Corollary 6.6, together with the results of Pink-Roessler [PR], provides an algebraic proof of another result of Ein-Lazarsfeld saying that if $P_1(X) = P_2(X) = 1$, then the Albanese map of X is surjective (cf. [EL] Theorem 4). This in turn implies the same for Kawamata's well-known theorem [Ka] saying that if $\kappa(X) = 0$, then the Albanese map is surjective. Indeed, in [PR] it is shown via the reduction mod p method of Deligne-Illusie, that the $V^j(\omega_X)$ are unions of translates of subtori of $\mathrm{Pic}^0(X)$. This allows the proof of [EL] Proposition 2.1 to go through unchanged, while Corollary 6.6 also shows that [EL] Proposition 2.2 has an algebraic proof.

7. Applications and examples for bundles on curves and Calabi-Yau fibrations

One of the main features of Theorem 3.7 is that it applies to essentially any integral transform. Here we exemplify with some statements for vector bundles on curves and on some threefold Calabi-Yau fibrations.

Semistable vector bundles on curves having a theta divisor. Let X be a smooth projective curve of genus $g \geq 2$, and let $SU_X(r, L)$ be the moduli space of semistable vector bundles on X of rank r and fixed determinant $L \in \mathrm{Pic}^d(X)$. The Picard group of $SU_X(r, L)$ is generated by the determinant line bundle $\mathcal{L}$. Results in this paper and the Strange Duality provide a Fourier-Mukai criterion for detecting base points of the linear series $|\mathcal{L}^k|$ for all k.

We start with a special case when this base locus is much better understood, namely the case of $d = r(g-1)$ and $k = 1$. In this case at least part of the criterion was already noted by Hein [He] (see below). It is well known (cf. [Be] §3) that if $d = r(g-1)$, then a semistable vector bundle E (or rather its S-equivalence class) is in the base locus of $|\mathcal{L}|$ if and only if

$$H^0(E \otimes \xi) \neq 0, \text{ for all } \xi \in \mathrm{Pic}^0(X).$$

Otherwise, the locus described set-theoretically as $\Theta_E := \{\xi \mid h^0(E \otimes \xi) \neq 0\} \subset \mathrm{Pic}^0(X)$ is a divisor, and one says that E *has a theta divisor*. We have the elementary:

Lemma 7.1. *If E is as above, then the following are equivalent:*
(1) E is not a base point for $|\mathcal{L}|$ (i.e. E has a theta divisor).
(2) E is a GV-sheaf with respect to any Poincaré bundle P on $X \times \mathrm{Pic}^0(X)$.
(3) $R^0\Phi_P E = 0$, i.e. E satisfies WIT_1 with respect to P.

Proof. The equivalence of (1) and (2) is the above discussion, plus the fact that since $\chi(E) = 0$, for all $\xi \in \mathrm{Pic}^0(X)$ we have $h^0(E \otimes \xi) = h^1(E \otimes \xi)$. By base change, the condition that E is not a base point is equivalent to the fact that $R^0\Phi_P E$ is supported on a proper subset of $\mathrm{Pic}^0(X)$, but since $R^0\Phi_P E$ is torsion-free[12], this is equivalent to requiring it to be 0. $\square$

Remark 7.2. Via an Abel-Jacobi embedding of X in its Jacobian $J(X)$, the functor $\mathbf{R}\Phi_P$ is the same as the Fourier-Mukai transform $\mathbf{R}\widehat{S}S$ on $J(X)$ applied to objects supported on X. Hence in the statement here we might as well talk about $\mathbf{R}\widehat{S}S$ instead of $\mathbf{R}\Phi_P$.

Fix now any polarization Θ on $J(X)$, and consider for any m the Fourier-Mukai transform

$$E_m := \mathbf{R}\Psi_P \mathcal{O}_{J(X)}(-m\Theta)[g] = \widehat{\mathcal{O}_{J(X)}(-m\Theta)},$$

which by base change is a vector bundle on X, usually called a *Raynaud bundle*. Using Lemma 7.1, together with the general statements of Corollary 3.12 and Lemma 2.5, we obtain the following criterion for detecting base points for the determinant line bundle. The equivalence of (1) and (3) has already been noted by Hein [He], Theorem 2.5. With a more careful study, he gives an effective bound for m in (3) (cf. *loc.cit*, Theorem 3.7).

Corollary 7.3. *Let X be a smooth projective curve of genus $g \geq 2$ and E a vector bundle in $SU_X(r, L)$, with $L \in \mathrm{Pic}^{g-1}(X)$. Then the following are equivalent:*
(1) E is not a base point for the linear series $|\mathcal{L}|$.
(2) $H^i(E \otimes E_m) = 0$, for all $i > 0$ and all $m \gg 0$.
(3) $H^0(E \otimes E_m^\vee) = 0$ for all $m \gg 0$.

[12] For example embed E in $E(D)$ for some divisor D on X of very large degree and apply Φ_P to the inclusion.

The Strange Duality conjecture, proved recently by Belkale [Bel] and Marian-Oprea [MO], allows for an extension of this in the most general setting. Let $SU_X(r,L)$ be as above, with $L \in \mathrm{Pic}^d(X)$. Let $h := (r,d)$, and $r_0 := r/h$ and $d_0 := d/h$. A general bundle F in the moduli space $U_X(kr_0, k(r_0(g-1)-d_0))$ (with arbitrary determinant) gives a generalized theta divisor

$$\Theta_F := \{E \mid h^0(E \otimes F) \neq 0\} \subset SU_X(r,L)$$

which belongs to the linear series $|\mathcal{L}^k|$ (cf. [DN]). The Strange Duality is equivalent to the fact that the divisors Θ_F span this linear series as F varies. It is well known that there exists as cover M of $U_X(kr_0, k(r_0(g-1)-d_0))$, étale over the stable locus, such that there is a universal bundle $\mathcal{E}$ on $X \times M$. We consider the Fourier-Mukai correspondence $\mathbf{R}\Phi_{\mathcal{E}} : \mathbf{D}(X) \to \mathbf{D}(M)$.[13] We thus obtain as before:

Lemma 7.4. *If E corresponds to a point in $SU_X(r,L)$, the following are equivalent:*
(1) E is not a base point for $|\mathcal{L}^k|$.
(2)E has a theta divisor $\Theta_E := \{F \mid h^0(E \otimes F) \neq 0\} \subset U_X(kr_0, k(r_0(g-1)-d_0))$.
(3) E is a GV-sheaf with respect to any universal bundle $\mathcal{E}$ on $X \times M$.
(4) $R^0\Phi_{\mathcal{E}}E = 0$, i.e. E satisfies WIT_1 with respect to $\mathcal{E}$.

We fix any generalized theta divisor on $U_X(kr_0, k(r_0(g-1)-d_0))$, and denote abusively by Θ its pullback to M. We consider for $m \gg 0$ the Fourier-Mukai transform

$$E_m^k := \mathbf{R}\Psi_{\mathcal{E}}\mathcal{O}_M(-m\Theta)[\dim M] = \widehat{\mathcal{O}_M(-m\Theta)},$$

which is a vector bundle on X generalizing Raynaud's bundles coming from the Jacobian. (This bundle can be constructed also as the push-forward of a Raynaud bundle on a spectral cover of C associated to the moduli space $U_X(kr_0, k(r_0(g-1)-d_0))$ as in [BNR].) As above, this provides the promised extension of Corollary 7.3.

Corollary 7.5. *Let X be a smooth projective curve of genus $g \geq 2$ and E a vector bundle in $SU_X(r,L)$, with $L \in \mathrm{Pic}^d(X)$. Then the following are equivalent:*
(1) E is not a base point for the linear series $|\mathcal{L}^k|$.
(2) $H^i(E \otimes E_m^k) = 0$, for all $i > 0$ and all $m \gg 0$.
(3) $H^0(E \otimes E_m^{k\vee}) = 0$ for all $m \gg 0$.

Relative moduli of sheaves on threefold Calabi-Yau fibrations. In theory one can study generic vanishing statements for any setting of the type: X is smooth projective, M is a fine moduli space of objects over X, and $\mathcal{E}$ is a universal object on $X \times M$ inducing the functor $\Phi_{\mathcal{E}}$. In practice, the main difficulty to be overcome is a good understanding of the vector bundles $\widehat{A^{-1}} = \mathbf{R}\Psi_{\mathcal{E}}(A^{-1})[\dim M]$, with A a very positive line bundle on M, on a case by case basis. Very few concrete examples seem to be known beyond the case of abelian varieties.[14] We would like to raise as a general problem to describe the structure of these vector bundles, given a specific moduli space.[15]

Here we give only a rather naive example of such a result for a Fourier-Mukai functor associated to threefolds with abelian or $K3$ fibration, considered first by Bridgeland and Maciocia in [BM] (cf. also [Br]). This works under special numerical hypotheses, based on results of

[13] For the statements we are interested in, this is a good as thinking that M is $U_X(kr_0, k(r_0(g-1)-d_0))$ itself, with the technical problem that as soon as $k \geq 2$ this moduli space will definitely not be fine.

[14] Besides curves or surfaces, where the vanishing of cohomology groups of appropriate semistable sheaves can usually be tested by hand – cf. for example §4, listing various examples of Yoshioka.

[15] For example, on a curve X, in the notation of the previous section the bundles E_m are well understood up to isogeny, due to Mukai's results on abelian varieties. How about the bundles E_m^k?

Yoshioka. The interested reader can prove similarly an analogous result in the case of elliptic threefolds.

Recall that a *Calabi-Yau fibration* is a morphism $\pi : X \to S$ of smooth projective varieties, with connected fibers, such that $K_X \cdot C = 0$ for all curves C contained in fibers of π. If it is of relative dimension at most two, then it is an elliptic, abelian surface, or $K3$-fibration (in the sense that the nonsingular fibers are of this type). Say π is flat, and consider a polarization H on X, and Y an irreducible component of the relative moduli space $M^{H,P}(X/S)$ of sheaves on X (over S), semistable with respect to H, and with fixed Hilbert polynomial P. The choice of P induces on every smooth fiber X_s invariants which are equivalent to the choice of a Mukai vector $v \in H^{ev}(X_s, \mathbb{Z})$ as in §4(5). Assuming that Y is also a threefold, and fine, Bridgeland and Maciocia (cf. [BM], Theorem 1.2) proved that it is smooth, and the induced morphism $\hat{\pi} : Y \to S$ is a Calabi-Yau fibration of the same type as π. In addition, if $\mathcal{E}$ is a universal sheaf on $X \times Y$, then the Fourier-Mukai functor $\mathbf{R}\Phi_{\mathcal{E}} : \mathbf{D}(X) \to \mathbf{D}(Y)$ is an equivalence of derived categories. We consider the following condition:

($\star$) For each $s \in S$ such that X_s is smooth, the Mukai vector v is primitive and isotropic, and the structure sheaf $\mathcal{O}_{X_s}$ satisfies WIT_2 with respect to the induced $\mathbf{R}\Phi_{\mathcal{E}_s} : \mathbf{D}(X_s) \to \mathbf{D}(Y_s)$.

Example 7.6. Papers of Yoshioka (e.g. [Yo2], [Yo3]) contain plenty of examples of surfaces where condition ($\star$) is satisfied. For some precise ones, both abelian and $K3$, cf. §4(5). Note that in all the cases we know, we have $\mathrm{Pic}(X_s) \cong \mathbb{Z}$.

If the first half of ($\star$) is satisfied, it is proved in [BM] §7 that the moduli space $M^{H,P}(X/S)$ does have a fine component Y which is a threefold, so the above applies. For simplicity we assume in the next statement that all the fibers of π are smooth, but please note Remark 7.8, which explains that the result can be made more general.

Proposition/Example 7.7. *Let X be a smooth projective threefold with a smooth Calabi-Yau fibration $\pi : X \to S$ of relative dimension two. Let H be a polarization on X and P a Hilbert polynomial, and assume that condition ($\star$) is satisfied. Consider a fine three-dimensional moduli space component $Y \subset M^{H,P}(X/S)$, and let $\mathcal{E}$ be a universal sheaf on $X \times Y$. Then ω_X is a GV_{-1}-sheaf with respect to $\mathcal{E}$. In particular*

$$H^i(X, \omega_X \otimes E) = 0, \ \forall \ i > 1, \ \forall \ E \in Y \text{ general.}$$

Proof. In order to prove that ω_X is GV_{-1} with respect to $\mathcal{E}$, it is enough to check condition (2) in Theorem 3.7. Given a very positive line bundle A on Y we want

$$H^i(\omega_X \otimes \widehat{A^{-1}}) = 0, \ \text{for all } i > 1, \tag{2}$$

where the Fourier transform is with respect to $\Psi_{\mathcal{E}}$.

We use the facts established in [BM]: the moduli space $M^{H,P}(X/S)$ restricts for each $s \in S$ to the corresponding moduli spaces of sheaves with Mukai vector v on X_s, stable with respect to the polarization $H_{|X_s}$. The functor $\mathbf{R}\Phi_{\mathcal{E}}$ is a relative Fourier-Mukai functor, which induces the respective fiberwise functors $\mathbf{R}\Phi_{\mathcal{E}_s} : \mathbf{D}(X_s) \to \mathbf{D}(Y_s)$. With our choice of polarization, each Y_s is a fine moduli space of sheaves on X_s, of the same dimension and in fact the same type as X_s (cf. [BM], §7.1, using Mukai's results).[16]

We check condition (2) by using the Leray spectral sequence for $\pi : X \to S$, namely

$$E_2^{i,j} := H^i(S, R^j\pi_*(\omega_X \otimes \widehat{A^{-1}})) \Rightarrow H^{i+j}(X, \omega_X \otimes \widehat{A^{-1}}).$$

[16]In fact each $\mathbf{R}\Phi_{\mathcal{E}_s}$ is an equivalence of derived categories.

For every $s \in S$ we have $(\omega_X \otimes \widehat{A^{-1}})_{|X_s} \cong \widehat{A_s^{-1}}$, where we denote $A_s := A_{|Y_s}$, and the transform on the right hand side is taken with respect to $\mathbf{R}\Psi_{\mathcal{E}_s}$.

By assumption we have that $\mathcal{O}_{X_s}$ satisfies WIT_2 with respect to $\mathbf{R}\Phi_{\mathcal{E}_s}$. We can then use Corollary 3.12 in a different direction ((3) $\Rightarrow$ (2)), to deduce that $H^i(X_s, \widehat{A_s^{-1}}) = 0$. Note that $\omega_{X|X_s} \cong \omega_{X_s} \cong \mathcal{O}_{X_s}$. Since π is smooth, we obtain by base-change that $R^j\pi_*(\omega_X \otimes \widehat{A^{-1}}) = 0$ for all $j \geq 1$. This immediately gives that $E_2^{i,j} = 0$ for $i \geq 2$ and all j, for $i = 1$ and $j \geq 1$, and also for $i = 0$ and $j = 2$. Thus the spectral sequence provides

$$H^i(X, \omega_X \otimes \widehat{A^{-1}}) = 0, \text{ for } i = 2, 3.$$

This is precisely (2), and we get that ω_X is GV_{-1}, or equivalently by Theorem A, that $R^i\Phi\mathcal{O}_X = 0$ for $i < 2$. $\square$

Remark 7.8. The same proof works in fact if we don't necessarily assume that π is smooth, but only that it is flat (a necessary assumption), plus the slightly technical condition $R^2\pi_*(\omega_X \otimes \widehat{A^{-1}}) = 0$ for A sufficiently positive on Y.[17] Indeed, since there is only a finite number of singular fibers, we obtain by base-change that $R^j\pi_*(\omega_X \otimes \widehat{A^{-1}})$ is supported on at most a finite set, for $j = 1, 2$, and it is of course 0 for $j \geq 3$. This immediately gives that $E_2^{i,j} = 0$ for $i \geq 2$ and all j, and also for $i = 1$ and $j \geq 1$. The only term which may cause trouble is $E_2^{0,2} = H^0(S, R^2\pi_*(\omega_X \otimes \widehat{A^{-1}}))$, and for its vanishing we have to use the assumption above.

References

[Ab] S. Abhyankar, Resolution of singularities of embedded surfaces, Academic Press, New York, 1966.

[Be] A. Beauville, Vector bundles on curves and generalized theta functions: recent results and open problems, in *Current topics in algebraic geometry*, Cambridge Univ. Press (1995), 17–33.

[BNR] A. Beauville, M.S. Narasimhan and S. Ramanan, Spectral curves and the generalized theta divisor, J. Reine Angew. Math. **398** (1989), 169–179.

[Bel] P. Belkale, The strange duality conjecture for generic curves, J. Amer. Math. Soc. **21** (2008), no.1, 235–258.

[BO] A. Bondal and D. Orlov, Semiorthogonal decomposition for algebraic varieties, preprint math.AG/9506012.

[Br] T. Bridgeland, Fourier-Mukai transforms for elliptic surfaces, J. reine angew. Math. **498** (1998), 115–133.

[BM] T. Bridgeland and A. Maciocia, Fourier-Mukai transforms for $K3$ and elliptic fibrations, J. Algebraic Geom. **11** (2002), no. 4, 629–657.

[BH] W. Bruns and J. Herzog, *Cohen-Macaulay rings*, Cambridge Univ. Press, 1993.

[Bu] N. Budur, Unitary local systems, multiplier ideals, and polynomial periodicity of Hodge numbers, Adv. Math. **221** (2009), no.1, 217–250.

[DI] P. Deligne and L. Illusie, Relèvements modulo p^2 et décomposition du complexe de de Rham, Invent. Math. **89** (1987), 247–270.

[DN] J.-M. Drezet and M.S. Narasimhan, Groupe de Picard des variétés des modules de fibrés semi-stables sur les courbes algébriques, Invent. Math. **97** (1989), 53–94.

[EGA III] A. Grothendieck and J. Dieudonné, Eléments de Géométrie Algébrique, III, Étude cohomologique des faisceaux coherents, Publ. Math. IHES **11** (1961) and **17** (1963).

[EL] L. Ein and R. Lazarsfeld, Singularities of theta divisors and the birational geometry of irregular varieties, J. Amer. Math. Soc. **10** (1997), 243–258.

[EV] H. Esnault and E. Viehweg, *Lectures on vanishing theorems*, DMV Seminar, 1992.

[GL1] M. Green and R. Lazarsfeld, Deformation theory, generic vanishing theorems, and some conjectures of Enriques, Catanese and Beauville, Invent. Math. **90** (1987), 389–407.

[GL2] M. Green and R. Lazarsfeld, Higher obstructions to deforming cohomology groups of line bundles, J. Amer. Math. Soc. **1** (1991), no.4, 87–103.

[Hac] Ch. Hacon, A derived category approach to generic vanishing, J. Reine Angew. Math. **575** (2004), 173–187.

[17] It should however be true that this holds most of the time, though admittedly we have not checked – at least in the case of abelian fibrations, after a relative isogeny one expects to reduce the calculation to a sheaf of the form $R^2\pi_*(\omega_X \otimes M)$ with M semiample, which would be 0 by Kollár's torsion-freeness theorem.

[HP] Ch. Hacon and R. Pardini, Birational characterization of products of curves of genus 2, Math. Res. Lett. **12** (2005), 129–140.

[Ha] R. Hartshorne, *Residues and duality*, Lecture Notes in Math. **20**, Springer-Verlag, 1966.

[He] G. Hein, Raynaud's vector bundles and base points of the generalized Theta divisor, Math. Z. **257** (2007), no. 3, 597–611.

[Iz] E. Izadi, Deforming curves representing multiples of the minimal class in Jacobians to non-Jacobians I, preprint mathAG/0103204.

[Ka] Y. Kawamata, Characterization of abelian varieties, Compositio Math. **43** (1981), 253–276.

[KMM] Y. Kawamata, K. Matsuda and K. Matsuki, Introduction to the minimal model problems, in Algebraic Geometry, Sendai 1985, Adv. Stud. Pure Math. **10**, North-Holland, Amsterdam (1987), 287–360.

[Ke] G. Kempf, Toward the inversion of abelian integrals II, Amer. J. of Math. **101** (1979), no.1, 184–202.

[Ko1] J. Kollár, Higher direct images of dualizing sheaves I, Ann. of Math. **123** (1986), 11–42.

[Ko2] J. Kollár, *Shafarevich maps and automorphic forms*, Princeton Univ. Press, 1995.

[KM] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, Cambridge Univ. Press, 1998.

[LB] H. Lange and Ch. Birkenhake, *Complex abelian varieties*, 2nd edition, Springer-Verlag 2004.

[La] R. Lazarsfeld, *Positivity in algebraic geometry* I & II, Ergebnisse der Mathematik und ihrer Grenzgebiete, vol. **48** & **49**, Springer-Verlag, Berlin, 2004.

[MO] A. Marian and D. Oprea, The rank-level duality for non-abelian theta functions, Invent. Math. **168** (2007), no.2, 225–247.

[Mo] S. Mori, Classification of higher-dimensional varieties, Algebraic Geometry, Bowdoin 1985, Proc. of Symp. in Pure Math **46** (1987), 269–331.

[Mou] Ch. Mourougane, Théorèmes d'annulation generique pour les fibrés vectoriels semi-négatifs, Bull. Soc. Math. France, **127** (1999), 115–133.

[Muk] S. Mukai, Duality between $D(X)$ and $D(\widehat{X})$ with its application to Picard sheaves, Nagoya Math. J. **81** (1981), 153–175.

[Mum] D. Mumford, *Abelian varieties*, Second edition, Oxford Univ. Press, 1974.

[Pa] G. Pareschi, Generic vanishing, Gaussian maps, and Fourier-Mukai transform, preprint math.AG/0310026.

[PP1] G. Pareschi and M. Popa, Regularity on abelian varieties I, J. Amer. Math. Soc. **16** (2003), 285–302.

[PP2] G. Pareschi and M. Popa, M-regularity and the Fourier-Mukai transform, Pure and Applied Math. Quarterly, F. Bogomolov issue II, **4** no.3 (2008).

[PP3] G. Pareschi and M. Popa, Regularity on abelian varieties III: relationship with Generic Vanishing and applications, preprint arXiv:0802.1021, to appear in the Proceedings of the Clay Mathematics Institute.

[PP4] G. Pareschi and M. Popa, Generic vanishing and minimal cohomology classes on abelian varieties, Math. Ann. **340** no.1 (2008), 209–222.

[PR] R. Pink and D. Roessler, A conjecture of Beauville and Catanese revisited, Math. Ann. **330** (2004), 293–308.

[Po1] A. Polishchuk, *Abelian varieties, theta functions and the Fourier transform*, Cambridge Univ. Press, 2002.

[Po2] A. Polishchuk, A_∞-structures, Brill-Noether loci and the Fourier-Mukai transform, Compos. Math. **140** (2004), 459–481.

[Yo1] K. Yoshioka, Chamber structure of polarizations and the moduli of stable sheaves on a ruled surface, Internat. J. Math. **7** (1996), 411–431.

[Yo2] K. Yoshioka, Some examples of isomorphisms induced by Fourier-Mukai functors, preprint math.AG/9902105.

[Yo3] K. Yoshioka, Stability and the Fourier-Mukai transform I, Math. Z. **245** (2003), 657–665.

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